SEQUENCES OF OPEN RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. We consider sequences of open Riemannian manifolds with boundary that have no regularity conditions on the boundary. To define a reasonable notion of a limit of such a sequence, we examine "δ inner regions" which avoid the boundary by a distance δ. We prove Gromov-Hausdorff compactness theorems for sequences of these "δ inner regions". We then build "glued limit spaces" out of the Gromov-Hausdorff limits of these δ interior regions and study the properties of these glued limit spaces. Our main applications assume the sequence is noncollapsing and has nonnegative Ricci curvature. We include open questions.

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1. Introduction

Recall that Gromov's Ricci Compactness Theorem states that a sequence of compact Riemannian manifolds with nonnegative Ricci curvature and a uniform upper bound on diameter has a subsequence which converges in the Gromov-Hausdorff sense to a metric space [7]. When the sequence of manifolds is noncollapsing, then the Gromov-Hausdorff limit spaces have a variety of proporties, particularly restrictions on their metrics, their Hausdorff measures, and their topologies. These properties were proven by Cheeger, Colding, Naber, Wei and the second author (c.f. [2], [3], [4] and [10]).

Here we consider an open Riemannian manifold, (M^m, g), endowed with the length metric, d_M, as in (4). We define the boundary to be

$$\partial M = \bar{M} \setminus M \tag{1}$$

where $\bar{M}$ is the metric completion of M. For example, (M^m, g) may be a smooth manifold with boundary. However, we do not require any smoothness conditions on this boundary.

First observe that Gromov's Ricci Compactness Theorem does not hold for precompact open manifolds with boundary that have a uniform upper bound on diameter even if they are flat and two dimensional:

Example 1.1. *The j-fold covering spaces, M_j, of the annulus, $Ann_0(1/j, 1) \subset \mathbb{E}^2$, depicted in Figure 1, are flat surfaces such that*

$$\mathrm{Diam}(M_j) \le 2 + \pi \textit{ and } \mathrm{Vol}(M_j) = j(\pi - \pi(1/j)^2). \tag{2}$$

See Remark 5.5 for the proof that there is no subsequence of these spaces with a Gromov-Hausdorff limit.

Figure 1. Models of Example 1.1: M_2, M_3, M_4...

Assuming both a uniform upper bound on volume and diameter, we still do not have Gromov-Hausdorff compactness:

Example 1.2. *The smooth regions, $M_j \subset \mathbb{E}^2$, with many splines depicted in Figure 2 have no subsequence with a Gromov-Hausdorff limit. See Example 2.13 for details.*

Figure 2. Models of Example 1.2: M_4, M_6, M_8, M_{12}...

In [11] and [9], Wong and Kodani have proven compactness theorems assuming curvature controls on the boundary. Since we do not wish to assume the boundary is smooth, we prove compactness theorems for regions which avoid the boundary [Theorem 1.4]. We then glue together the limits of these regions [Theorem 6.3 and prove these glued limit spaces have nice properties [Theorem 8.8].

Definition 1.3. *Given an open Riemannian manifold, (M, g_M), and a positive $\delta > 0$, we define the δ inner region as follows*

$$M^\delta = \left\{x \in M : \ d_M(x, \partial M) > \delta\right\} \tag{3}$$

where ∂M is defined as in (1),

$$d_M(x, y) := \inf\left\{L_g(C) : \ C : [0, 1] \to M, \ C(0) = x, \ C(1) = y\right\} \tag{4}$$

and

$$L_g(C) = \int_0^1 g(C'(t), C'(t))\, dt. \tag{5}$$

Note that there are two metrics on the δ interior region, M^δ: the restricted metric, d_M, and the induced length metric,

$$d_{M^\delta}(x, y) := \inf\left\{L_g(C) : \ C : [0, 1] \to M^\delta, \ C(0) = x, \ C(1) = y\right\}. \tag{6}$$

Note that d_{M^δ} is only defined between points in connected components of M^δ. The intrinsic diameter

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) = \sup\left\{d_{M^\delta}(x, y) : \ x, y \in M^\delta\right\}, \tag{7}$$

will be infinite if M^δ is not connected by rectifiable paths.

Theorem 1.4. *Given $m \in \mathbb{N}$, $\delta > 0$, $D > 0$, $V > 0$, and $\theta > 0$, set $\mathcal{M}_\theta^{m,\delta,D,V}$ to be the class of open Riemannian manifolds, M, with boundary of dimension m, with nonnegative Ricci curvature, $\mathrm{Vol}(M) \le V$, and*

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) \le D, \tag{8}$$

that are noncollapsing at a point,

$$\exists q \in M^\delta \text{ such that } \mathrm{Vol}(B_q(\delta)) \ge \theta\delta^m. \tag{9}$$

If $(M_j, g_j) \subset \mathcal{M}_\theta^{m,\delta,D,V}$, then there is a subsequence, $(M^\delta_{j_k}, d_{M_{j_k}})$, such that the metric completions with the restricted metric, d_{M_j}, converge in the Gromov-Hausdorff sense to a metric space, (Y^δ, d).

Example 1.2 satisfies the conditions of this theorem, demonstrating why we can only obtain Gromov-Hausdorff convergence of the M_j^δ instead of the M_j themselves. The M_j^δ of Example 1.1 do not have Gromov-Hausdorff converging subsequences (see Remark 5.5) demonstrating the necessity of the hypothesis requiring an upper volume bound. In Theorem 5.2, stated within, we remove the intrinsic diameter condition, (8), and the noncollapsing condition, (200), and assume conditions on closed geodesics and constant sectional curvature instead.

Theorem 1.4 and Theorem 5.2 are proven in Section 5. First we review of Gromov-Hausdorff convergence in Section 2. In Sections 3 and 4 we study the limits of inner regions in sequences of manifolds that have Gromov-Hausdorff limits. See in particular Theorem 4.1. These sections contain many examples.

In Section 6 we define glued limit spaces for any sequence of open Riemannian manifolds, (M_j, g_j) assuming that for all $\delta > 0$, (M_j^δ, d_j) converge in the Gromov-Hausdorff sense to a metric space (Y^δ, d_δ). We build a "glued limit space", (Y, d_Y), from these Y^δ in Theorem 6.1 and Theorem 6.3. Note that this glued limit space may exist even when (M_j, d_j) has no Gromov-Hausdorff limit as in Examples 2.13, [Remark 6.10]. The glued limit may not be precompact even when one has a sequence of flat Riemannian manifolds with boundary [Examples 6.11 and 6.12].

In general the glued limit space of a sequence of M_j need not be unique [Example 6.16]. However, if the (M_j, d_{M_j}) have a Gromov-Hausdorff limit, (X, d_X),then the glued limit space is unique and is embedded isometrically into X [Remark 6.7]. Example 4.10 demonstrates that these limits need not be isometric even when the (M_j, g_j) are regions in the Euclidean plane satisfying all the hypothesis of Theorem 1.4 [Remark 6.8]. Intuitively, regions which collapse relative to the boundary disappear while regions which collapse that lie far from the boundary, need not disappear.

In Section 7 we apply Theorems 5.2 and 1.4 to construct glued limit spaces for sequences of manifolds with curvature bounds [Theorems 7.1 and 7.4. In Section 8 we explore the properties of these glued limit spaces. First we present an example where the curvature bounds in the sequence of manifolds is lost in the Gromov-Hausdorff limit [Example 8.1]. Then we prove Proposition 8.4 concerning glued limits of manifolds with constant sectional curvature. We close with Theorem 8.8, proving that glued limits constructed under the conditions of Theorem 1.4 have Hausdorff dimension m, Hausdorff measure $\leq V$, and positive density everywhere. This final theorem is proven using Theorem 8.3 which proves certain balls in glued limit spaces are the Gromov-Hausdorff limits of nice balls in the open manifolds, combined with the Bishop-Gromov Volume Comparison Theorem [7] and Colding's Volume Convergence Theorem [3].

Throughout the paper we state open questions: Question 6.14, Question 8.6, Question 8.7, Question 8.10, and Question 8.9. The first author is in the process of proving Question 8.10 as part of her doctoral dissertation. Please contact us if you would like to work on one of the other open questions or if you are interested in extending our theorems to the setting where the sequence has a negative uniform lower Ricci curvature bound or is allowed to collapse.

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2. Background

Here we review Gromov-Hausdorff convergence and Gromov's Compactness Theorem [7]. A good resource for this material is [1].

2.1. **Hausdorff Convergence.** In [7], Gromov defined the Gromov-Hausdorff distance between pairs of compact metric spaces. We review this definition here.

Definition 2.1 (Hausdorff)**.** *The Hausdorff distance between two compact subsets, A_1, A_2, of a metric space, Z, with metric, d_Z, is defined*

$$d_H^Z(A_1, A_2) = \inf\left\{r : A_1 \subset T_r(A_2),\ A_2 \subset T_r(A_1)\right\} \tag{10}$$

where the tubular neighborhood, $T_r(A) = \{x \in Z : d_Z(x, A) < r\}$.

Observe that if one has a sequence of compact subsets $A_j \subset Z$ such that $d_H(A_j, A_\infty) \to 0$, then for all $a \in A_\infty$ there exists $a_j \in A_j$ such that $\lim_{j\to\infty} a_j = a$. One also has the following lemma:

Lemma 2.2. *Suppose $A_\infty \subset Z$ are compact, $d_H^Z(A_j, A_\infty) = h_j \to 0$ and $a_j \in A_j$ such that $d_Z(a_j, a_\infty) = \delta_j \to 0$. Then for all $r > 0$ there exists $r_j = r + \delta_j + h_j \to r$ such that the closed balls converge*

$$d_H\left(\bar{B}_{a_j}(r_j) \cap A_j, \bar{B}_{a_\infty}(r) \cap A_\infty\right) \to 0. \tag{11}$$

Here we are not assuming A_∞ or A_j are length spaces. For completeness of exposition we include the proof of this well known lemma:

Proof. Suppose $x \in \bar{B}_{a_\infty}(r) \cap A_\infty$, then $d_Z(x, a_\infty) \le r$ and $x \in A_\infty \subset T_{h_j}(A_j)$. So there exists $y_j \in A_j$ such that $d_Z(x, y_j) < h_j$. By triangle inequality,

$$d(y_j, a_j) \le d(y_j, x) + d(x, a_\infty) + d(a_\infty, a_j) \le h_j + r + \delta_j = r_j. \tag{12}$$

Thus

$$\bar{B}_{a_\infty}(r) \cap A_\infty \subset T_{h_j}(\bar{B}_{a_j}(r_j) \cap A_j). \tag{13}$$

Now we need only show there exists $\varepsilon_j \to 0$ such that

$$\bar{B}_{a_j}(r_j) \cap A_j \subset T_{\varepsilon_j}(\bar{B}_{a_\infty}(r) \cap A_\infty). \tag{14}$$

Suppose not. Then there exists $\varepsilon_0 > 0$ such that for all j sufficiently large, there is an

$$x_j \in \left(\bar{B}_{a_j}(r_j) \cap A_j\right) \setminus T_{\varepsilon_0}\left(\bar{B}_{a_\infty}(r) \cap A_\infty\right). \tag{15}$$

Since Z is compact and $T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty)$ is open, a subsequence of the x_j converge to some

$$x_\infty \notin T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty). \tag{16}$$

Since $d(x_j, a_j) \le r_j$, we have $d(x_\infty, a_\infty) \le r$. Since $x_j \in A_j$, there exists $y_j \in A_\infty$ such that $d(x_j, y_j) < h_j$. By the triangle inequality

$$y_j \in B_{a_\infty}(r + h_j) \cap A_\infty. \tag{17}$$

Observe that for our subsequence $y_j \to x_\infty$, thus

$$x_\infty \in \bar{B}_{a_\infty}(r) \cap A_\infty \subset T_{\varepsilon_0}(\bar{B}_{a_\infty}(r) \cap A_\infty) \tag{18}$$

which is a contradiction. □

2.2. **Gromov-Hausdorff Convergence.**

Definition 2.3. *An isometric embedding, $\varphi : (X, d_X) \to (Z, d_Z)$ between metric spaces is a mapping which preserves distances:*

$$d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \tag{19}$$

Definition 2.4 (Gromov)**.** *The Gromov-Hausdorff distance between a pair of compact metric spaces, (X_1, d_{X_1}) and (X_2, d_{X_2}) is defined*

$$d_{GH}\Big((X_1, d_{X_1}), (X_2, d_{X_2})\Big) = \inf\Big\{d_Z(\varphi_1(X_1), \varphi_2(X_2)) : \ \varphi_i : X_i \to Z\Big\} \tag{20}$$

where the infimum is taken over all isometric embeddings $\varphi_i : X_i \to Z$ and all metric spaces, Z.

Gromov proved that the Gromov-Hausdorff distance is a distance on the space of compact metric spaces. When studying metric spaces, X_i, which are only precompact, one takes the metric completions, $\bar{X}_i$, before comparing such spaces using the Gromov-Hausdorff distance:

Definition 2.5. *Given a precompact metric space space, (X, d_X), the metric completion, $(\bar{X}, d_X)$, consists of equivalence classes of Cauchy sequences, $\{x_1, x_2, x_3, ...\}$, in X, where*

$$d_X(\{x_j\}, \{y_j\}) = \lim_{j\to\infty} d_X(x_j, y_j) \tag{21}$$

and two Cauchy sequences are equivalent if the distance between them is 0. There is an isometric embedding

$$\varphi : X \to \bar{X} \text{ such that } \varphi(x) = \{x, x, x, ...\}. \tag{22}$$

In this paper we define the boundary of an open metric space

$$\partial X = \bar{X} \setminus X. \tag{23}$$

When M is a smooth Riemannian manifold with boundary, then this notion of boundary agrees with the standard notion of boundary. However, if M is a smooth Riemannian manifold with a singular point removed, then the boundary in our setting is just the missing singular point.

2.3. **Lattices and Gromov-Hausdorff Convergence.** One technique that can be applied to produce amazingly complicated Gromov-Hausdorff limits from surfaces, is to construct lattices. The basic well known lemma is as follows:

Lemma 2.6. *Let $X = [a_1, b_1] \times \cdots \times [a_k, b_k]$ with the taxi product metric*

$$d_X((x_1, ...x_k), (y_1, ...y_k)) = \sum_{i=1}^{k} |x_i - y_i|. \tag{24}$$

Then for any $\varepsilon > 0$ there exists a 2 dimensional manifold M_ε such that

$$d_{GH}(M_\varepsilon, X) < \varepsilon. \tag{25}$$

The classic application of this lemma is to construct a Gromov-Hausdorff limit of Riemannian surfaces which is infinite dimensional:

Example 2.7. *Let $X_j = [0, 1] \times [0, 1/2] \times \cdots \times [0, 1/2^j]$ with the taxi metric and let*

$$X = [0, 1] \times [0, 1/2] \times \cdots \times [0, 1/2^j] \times \cdots \tag{26}$$

be the infinite dimensional space also with the taxi metric:

$$d_X((x_1, x_2, ...), (y_1, y_2, ...)) = \sum_{i=1}^{\infty} |x_i - y_i|. \tag{27}$$

Then

$$d_{GH}(X_k, X) \leq \sum_{j=k+1}^{\infty} 1/2^j = 1/2^k \to 0. \tag{28}$$

Thus by Lemma 2.6 we have a sequence of surfaces M_k converging to X as well.

Since we are interested in manifolds with boundary, we will prove a stronger version of Lemma 2.6 that can be applied to produce examples later in the paper.

Proposition 2.8. *Suppose $X = [a_1, b_1] \times \cdots \times [a_k, b_k]$ with the taxi product metric and $A \subset \partial X$ (possibly empty), then for any $\varepsilon > 0$ there exists an open Riemannian surface, M, with boundary, ∂M (possible empty) such that*

$$d_{GH}(M, X) < \varepsilon \text{ and } d_{GH}(\partial M, A) < \varepsilon. \tag{29}$$

We can also prove that if we have a collection of X_k and $A_k \subset \partial X_k$ as above with subsets $B_k \subset X_k$ and isometric embeddings $\psi_k : B_{k+1} \to B_k$, and we glue together $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_k$ along these isometric embeddings, and set $A = \cup A_k \subset X$, then for any $\varepsilon > 0$ we have an open Riemannian surface, M, with boundary, ∂M (possible empty) such that

$$d_{GH}(M, X) < \varepsilon \text{ and } d_{GH}(\partial M, A) < \varepsilon. \tag{30}$$

In fact, for any $\delta > 0$, using the restricted distances, we have

$$d_{GH}((M \setminus T_\delta(\partial M), d_M), (X \setminus T_\delta(A), d_X)) < \varepsilon. \tag{31}$$

Proof. For the first part, we take a lattice $Y'_\varepsilon \subset Y_\varepsilon \subset X$ such that $X \subset T_{\varepsilon/2}(Y_\varepsilon)$. Here we use Y'_ε to denote the points and Y_ε to include 1 dimensional edges between the points in the lattice. Observe that $d_{Y_\varepsilon}(y_1, y_2) = d_X(y_1, y_2)$ because we are using the taxi norm. Let $A_\varepsilon \in Y'_\varepsilon$ be chosen such that $A_\varepsilon \subset T_{\varepsilon/2}(A)$. So

$$d_{GH}(Y_\varepsilon, X) < \varepsilon/2 \text{ and } d_{GH}(A_\varepsilon, A) < \varepsilon/2. \tag{32}$$

Note that we may now view Y_ε as a graph. For example, if $X = [0, 5] \times [0, 6]$ and $A = [0, 5] \times \{6\}$ and $\epsilon = 1$, then the left side of Figure 3 is the graph Y_ε with A_ε is depicted in red.

Next we construct a smooth surface M by replacing the lattice points in $A_\varepsilon \subset Y'_\varepsilon$ by small hemispheres of diameter $<< \varepsilon$ and lattice points in $Y'_\varepsilon \setminus A_\varepsilon$ by small spheres of diameter $<< \varepsilon$. We replace the line segments in Y_ε by arbitrarily thin cylinders of the same length, small enough that we can glue them to their corresponding spheres smoothly replacing disjoint balls in those spheres or hemispheres. This creates a smooth manifold, M, such that ∂M is a union of the boundaries of the hemispheres such that

$$d_{GH}(Y_\varepsilon, M) < \varepsilon/2 \text{ and } d_{GH}(A_\varepsilon, \partial M) < \varepsilon/2. \tag{33}$$

See the right side of Figure 3, where M^2 is depicted in black and ∂M^2 is in red. This completes the first claim in the proposition.

To complete the rest, we take M_k consisting of tubes joined at spheres and hemispheres close to X_k as above such that

$$d_{GH}(X_k, M_k) < \varepsilon/k \text{ and } d_{GH}(A_k, \partial M_k) < \varepsilon/k. \tag{34}$$

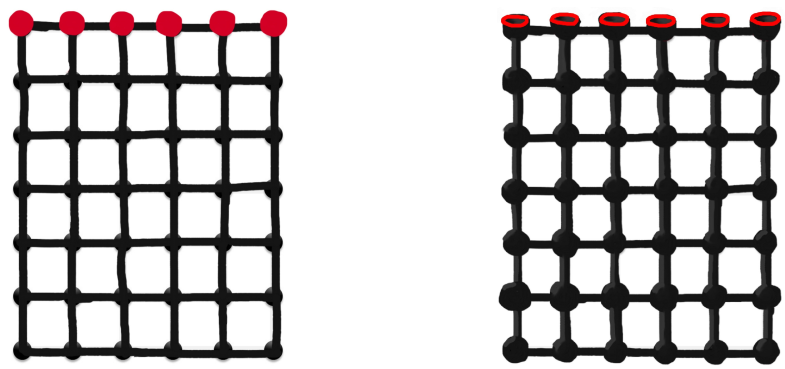

FIGURE 3. $A_\varepsilon \subset Y_\varepsilon$ and $\partial M \subset M$ as in the proof of Proposition 2.8.
Note that these graphics will be improved before publication.

Note that in the construction above we could have created $B'_k \subset Y'_k$ corresponding to B_k. We have $\varepsilon/(2k)$ almost distance preserving maps $\psi'_k : B'_{k+1} \to B'_k$. So now we glue together the M_k to form M as follows. If $b \in B'_k$ maps to $\psi'_k(b) \in B'_k$ we connect the sphere or hemisphere corresponding to b in M_k to a sphere or hemisphere corresponding to $\psi_k(b)$ in M_{k+1} by a very short, very thin tube. □

2.4. **Review of Gromov's Compactness Theorem.** In [7], Gromov proved a compactness theorem for sequences of compact metric spaces. We review this theorems and related propositions here.

Theorem 2.9. *[Gromov] Given a $D > 0$ and a function $N : (0, D] \to \mathbb{N}$, we define the collection, $\mathcal{M}^{D,N}$ of compact metric spaces, (X, d_X) with diameter $\leq D$ that can be covered by $N(\epsilon)$ balls of radius $\epsilon > 0$:*

$$X \subset \bigcup_{i=1}^{N(\epsilon)} B_{x_i}(\epsilon). \tag{35}$$

This collection $\mathcal{M}^{D,N}$ is compact with respect to the Gromov-Hausdorff distance.

It is standard to determine whether a metric space lies in such a compact collection by examining maximal collections of disjoint balls:

Proposition 2.10. *Given a metric space (X, d_X). Let N be the maximum number of pairwise disjoint balls of radius $\epsilon/2$ that can lie in X. Then the minimum number of balls of radius ϵ required to cover X is $\leq N$.*

Proof. Let $\{B_{x_i}(\epsilon/2) : i = 1, ..., N\}$ be a maximal collection of pairwise disjoint balls of radius $\epsilon/2$. Let $x \in X$. Then $\exists i \in \{1, ..., N\}$ such that $B_{x_i}(\epsilon/2) \cap B_x(\epsilon/2) \neq \emptyset$. Thus $d_X(x, x_i) < \epsilon$ and

$$X \subset \bigcup_{i=1}^{N} B_{x_i}(\epsilon). \tag{36}$$

□

In a Riemannian manifold or metric measure space, the volumes of balls may thus be applied to determine the function, N.

Proposition 2.11. *If there exists* $\Theta > 0$ *such that*

$$\operatorname{Vol}(B_p(\epsilon))/\operatorname{Vol}(M) \geq \Theta \tag{37}$$

then the maximum number of disjoint balls of radius ϵ *is* $\leq 1/\Theta$.

Proof.

$$\operatorname{Vol}(M) \geq \sum_{i=1}^{N} \operatorname{Vol}(B_{x_i}(\epsilon)) \geq \sum_{i=1}^{N} \Theta \operatorname{Vol}(M) = N\Theta \operatorname{Vol}(M). \tag{38}$$

□

Gromov applies his compactness theorem in conjunction with these propositions to study the compactness of sequences of compact Riemannian manifolds for which one is able to control the volumes of balls. We will apply the same idea to study sequences of metric completions of open manifolds.

One of the beauties of Gromov's Compactness Theorem, is that he has proven the converse as well:

Theorem 2.12. *[Gromov] Suppose* (X_j, d_j) *are compact metric spaces. Suppose that there exists* $\epsilon_0 > 0$ *such that* X_j *contains at least* j *disjoint balls of radius* ϵ_0. *Then no subsequence of the* X_j *has a Gromov-Hausdorff limit.*

In particular, if $(X_j, d_{X_j}) \xrightarrow{\text{GH}} (X, d_X)$ then they have a uniform upper bound on diameter. Nor can they have many splines, as in the following example:

Example 2.13. *Let*

$$M_j = \{(\theta, r) : \ \theta \in S^1, \ r \in (1, 3 + \cos(j\theta))\} \tag{39}$$

with metric $g_j = dr^2 + r^2 d\theta^2$. *Then* $\operatorname{Vol}(M_j) \leq \pi 4^2$ *and* $\operatorname{Diam}(M_j) \leq 3 + \pi + 3$ *with* 0 *sectional curvature.*

Observe that in M_j, *the balls of radius* 1 *about* $(2\pi k/j, 3)$ *are disjoint because paths between these points in* M_j *must reach within* $r \leq 2$ *between the splines and so have length* $\geq 2(3 - 2)$. *Thus there are* j *disjoint balls of radius* 1 *in* M_j *and no subsequence of the metric completions of* M_j *converge in the Gromov-Hausdorff sense.*

Example 2.14. *Let*

$$X_j = ([0,1] \times [0,1]) \sqcup ([0,1] \times [0,1/2]) \sqcup \cdots \sqcup ([0,1] \times [0,1/2^j]) \tag{40}$$

be a disjoint union of spaces with taxicab metrics glued with a gluing map $\psi(0, y) = (0, y)$. *Then* X_j *has no Gromov-Hausdorff converging subsequence because it has* j *disjoint balls of radius* 1 *about points* $(1, 0)$. *If we take surfaces* M_j *as constructed in Proposition 2.8, such that*

$$d_{GH}(M_j, X_j) \to 0, \tag{41}$$

they also have no Gromov-Hausdorff converging subsequence.

Defining an appropriate compact metric space and applying Theorem 2.16, in a later paper, [6](page 65), Gromov proved the following useful theorem.

Theorem 2.15 (Gromov)**.** *If one has a sequence of compact metric spaces,* (X_j, d_{X_j}), *such that* $(X_j, d_{X_j}) \xrightarrow{GH} (X_\infty, d_{X_\infty})$, *then there exists a common compact metric space* Z *and isometric embeddings* $\varphi_j : (X_j, d_{X_j}) \to (Z, d_Z)$ *such that* $d_H(\varphi_j(X_j), \varphi_\infty(X_\infty)) \to 0$.

Theorem 2.16 (Blaschke)**.** *If Z is a compact metric space then every sequence of closed subsets of Z has a subsequence that converges in Hausdorff sense to a closed subset.*

Theorem 2.15 implies the Gromov-Hausdorff Arzela-Ascoli Theorem:

Theorem 2.17 (Gromov)**.** *If $X_j \xrightarrow{GH} X$ and $Y_j \xrightarrow{GH} Y$ and $f_j : X_j \to Y_j$ are equicontinuous,*

$$\forall \epsilon > 0\, \exists \delta_\epsilon > 0 \text{ such that } d_{X_j}(p,q) < \delta_\epsilon \implies d_{Y_j}(f_j(p), f_j(q)) < \epsilon, \tag{42}$$

then there is a subsequence with a continuous limit function

$$f : X \to Y. \tag{43}$$

If the f_j are isometric embeddings, then so is f.

In particular, if the X_j are geodesic spaces, then so is the limit space [7].

2.5. **Gromov's Ricci Compactness Theorem.** In this section we review Gromov's Ricci Compactness Theorem which is based on the Bishop-Gromov Volume Comparison Theorem [7]:

Theorem 2.18 (Bishop-Gromov)**.** *If M is an m dimensional Riemannian manifold with boundary that has nonnegative Ricci curvature and $B_p(R) \subset M^m$ does not reach the boundary, then for all $r \in (0, R)$ we have*

$$\frac{\operatorname{Vol}(B_p(r))}{\operatorname{Vol}(B_p(R))} \geq \left(\frac{r}{R}\right)^m \tag{44}$$

Gromov's Ricci Compactness Theorem was originally stated for compact manifolds without boundary:

Theorem 2.19 (Gromov)**.** *Let $m \in \mathbb{N}$, $D > 0$ and let $\mathcal{M}^{m,D}$ be the class of compact m dimensional Riemannian manifolds, M, with nonnegative Ricci curvature and $\operatorname{Diam}(M) \leq D$. Here the manifolds do not have boundary. Then $\mathcal{M}^{m,D}$ is precompact with respect to the Gromov-Hausdorff distance.*

In fact, Gromov's Compactness Theorem has a commonly used version applied to balls which we state as follows:

Theorem 2.20 (Gromov)**.** *Let $m \in \mathbb{N}$, $D > 0$ and let $\mathcal{M}^m$ be the class of compact m dimensional Riemannian manifolds, M, with nonnegative Ricci curvature. Suppose $M_j \in \mathcal{M}^m$ and suppose $B_{p_j}(D)$ do not reach the boundary, then there exists a subsequence such that $(B_{p_j}(D/3), d_{M_j})$ converges in the Gromov-Hausdorff distance.*

For completeness of exposition we show how Gromov's original proof implies Theorem 2.20.

Proof. Let $q \in B_{p_j}(D/3)$. Then

$$B_{p_j}(D/3) \subset B_q(2D/3) \subset B_{p_j}(D) \tag{45}$$

does not reach the boundary of M_j, so we may apply the Bishop-Gromov Volume Comparison Theorem to see that:

$$\frac{\operatorname{Vol}(B_q(r))}{\operatorname{Vol}(B_{p_j}(D/3))} \geq \frac{r^m}{(2D/3)^m}\frac{\operatorname{Vol}(B_q(2D/3))}{\operatorname{Vol}(B_{p_j}(D/3))} \tag{46}$$

$$\geq \frac{r^m}{(2D/3)^m}\frac{\operatorname{Vol}(B_{p_j}(D/3))}{\operatorname{Vol}(B_{p_j}(D/3))} = \frac{(3r)^m}{(2D)^m}. \tag{47}$$

So now we may apply Proposition 2.11 to complete the proof. ◻

2.6. **Volume Convergence Theorems.** In [3], Colding proved the following volume convergence theorem:

Theorem 2.21 (Colding)**.** *Let M_j^m be complete Riemannian manifolds with nonnegative Ricci curvature and $p_j \in M_j$ such that*

$$B_{p_j}(1) \xrightarrow{GH} B_0(1) \subset \mathbb{E}^m \tag{48}$$

where $\mathbb{E}^m$ is Euclidean space of dimension m, then

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(1)) = \mathrm{Vol}(B_0(1)). \tag{49}$$

Remark 2.22. *The proof of this theorem does not in fact require global nonnegative Ricci curvature on a complete manifold. In fact M_j^m could be an open manifold as long as $B_{p_j}(2) \subset \bar{M}_j^m$ does not hit the boundary. In fact one may not even need a radius of* 2*.*

Colding applied this theorem to prove a number of theorems including one in which the Gromov-Hausdorff limit is an arbitrary compact Riemannian manifold of the same dimension (also [3]):

Theorem 2.23 (Colding)**.** *Let M_j^m and M_∞^m be compact Riemannian manifolds with nonnegative Ricci curvature for $j = 1, 2, 3, ...$ such that*

$$M_j^m \xrightarrow{GH} M_\infty^m. \tag{50}$$

Then for all $r > 0$ and for all $p_j \in M_j$ such that $p_j \to p_\infty$ we have

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(r)) = \mathrm{Vol}(B_{p_\infty}(r)). \tag{51}$$

Remark 2.24. *Again Colding's proof does not really require M_j to be complete. These M_j could be open Riemannian manifolds as long as $B_{p_j}(r) \subset \bar{M}_j$ does not hit the boundary. Here we do not need to worry about twice the radius because the proof involves estimating countable collections of small balls $B_{q_{j,i}}(\epsilon_{j,i})$ in $B_{p_j}(r)$ and applying Theorem 2.21 to those small balls and one can always ensure the $B_{q_{j,i}}(2\epsilon_{j,i})$ avoid the boundary as in Remark 2.22*

Cheeger-Colding then conducted a study of the properties of Gromov-Hausdorff limits of manifolds of nonnegative Ricci curvature in [2]. They improve upon Theorem 2.23, allowing M_∞ to be an arbitrary limit space as long as the sequence is noncollapsing:

Theorem 2.25 (Cheeger-Colding)**.** *Let $V_0 > 0$ and let M_j^m be compact Riemannian manifolds with nonnegative Ricci curvature for $j = 1, 2, 3, ...,$ such that*

$$M_j^m \xrightarrow{GH} M_\infty^m \text{ and } \mathrm{Vol}(M_j^m) \geq V_0. \tag{52}$$

Then for all $r > 0$ and for all $p_j \in M_j$ such that $p_j \to p_\infty \in M_\infty$ we have

$$\lim_{j\to\infty} \mathrm{Vol}(B_{p_j}(r)) = \mathcal{H}^m(B_{p_\infty}(r)) \tag{53}$$

where $\mathcal{H}^m$ is the Hausdorff measure of dimension m.

Remark 2.26. *Again this theorem is proven locally, so as in Remark 2.24 this theorem holds when M_j^m are open Riemannian manifolds as long as $B_{p_j}(r) \subset \bar{M}_j^m$ do not touch the boundary.*

Of course, Cheeger and Colding study more than just manifolds with nonnegative Ricci curvature and more than just noncollapsing sequences in their work, but these theorems are the only ones needed in this paper. See also work of the second author with Wei for an adaption of their volume convergence theorem which deals with Hausdorff measures defined using restricted vs intrinsic distances [10].

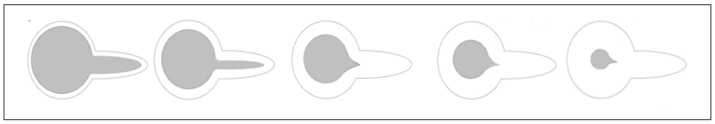

FIGURE 4. Example 3.1: Single M varying δ

3. PROPERTIES OF INNER REGIONS

Given an open Riemannian manifold, M, we have defined the δ inner region, M^δ in Definition 1.3. Note that these spaces are open Riemannian manifolds, however we will study them using the restricted distance, d_M, rather than the intrinsic length metric, d_{M^δ}, defined in (6). There are natural isometric embeddings of (M^δ, d_M) and its metric completion $(\bar{M}^\delta, d_M)$ into (M, d_M). Thus the metric completion is, in fact, compact when M is precompact. This occurs, for example, when M has finite diameter.

Example 3.1. *In Figure 3, we depict a single flat manifold, M^2, which is a flat disk with a spline attached. For a sequence of $\delta_1 < \delta_2 < \delta_3 < \delta_4$, the grey inner regions depict M^{δ_i}. For δ sufficiently large M^δ is an empty set.*

Lemma 3.2. *For any sequence $\delta_i \to 0$, we have*

$$M = \bigcup_{i=1}^{\infty} M^{\delta_i}. \tag{54}$$

In fact,

$$M = \bigcup_{\delta>0} M^{\delta}. \tag{55}$$

Proof. Let $x \in M$, since M is open $\varepsilon = d_M(x, \partial M) > 0$. Then $x \in M^{\varepsilon/2}$. □

Lemma 3.3. *Let $\delta > \delta' > 0$. If $y \in M^\delta$ then for any $\varepsilon < \delta - \delta'$ we have*

$$B(y, \varepsilon) = \{x \in M : \; d_M(x, y) < \varepsilon\} \subset M^{\delta'}. \tag{56}$$

Proof. Let $x \in B(y, \varepsilon)$, so $d_M(x, y) < \delta - \delta'$. Since $y \in M^\delta$, for all $z \in \partial M$ $d_M(y, z) > \delta$. By the triangle inequality,

$$d_M(x, z) \geq d_M(y, z) - d_M(x, y) > \delta - (\delta - \delta') = \delta'. \tag{57}$$

□

Inner regions, M^δ, with restricted metrics, d_M, are not necessarily length spaces.

Example 3.4. *In the flat open manifold*

$$M = \{(x, y) : x^2 + y^2 \in (1, 25)\} \subset \mathbb{E}^2 \tag{58}$$

the distance between $(3, 1)$ *and* $(-3, 1)$ *is*

$$d_M((3, 1), (-3, 1)) = 6 \tag{59}$$

because they are joined by curves of length arbitrarily close to 6. *However for* $\delta = 1$ *we have*

$$M^\delta = \{(x, y) : x^2 + y^2 \in (4, 16)\} \subset \mathbb{E}^2. \tag{60}$$

The length of any curve in M^δ between $(3, 1)$ *and* $(-3, 1)$ *must go around* $(0, 2)$ *and thus has length at least* $2\sqrt{9+1} > 6$.

In fact inner regions of path connected manifolds need not be connected:

Example 3.5. *Let our manifold be the connected union of balls in the Euclidean plane:*

$$M = B_{(4,0)}(5) \cup B_{(-4,0)}(5) \subset \mathbb{E}^2 \tag{61}$$

Then

$$\partial M = A_+ \cup A_- \tag{62}$$

where

$$A_+ = \partial B_{(4,0)}(5) \cap \{(x, y) : x \geq 0\} \tag{63}$$

$$A_- = \partial B_{(-4,0)}(5) \cap \{(x, y) : -x \geq 0\} \tag{64}$$

Note that

$$(0, 3), (0, -3) \in \partial M. \tag{65}$$

Thus for $\delta > 3$,

$$M^\delta \cap \{(0, y) : y \in \mathbb{R}\} = \emptyset. \tag{66}$$

However for $\delta < 5$, we have

$$(4, 0), (-4, 0) \in M^\delta. \tag{67}$$

Thus M^δ is not connected for $\delta \in (3, 5)$.

4. Manifolds with Gromov-Haudorff limits have Converging Inner Regions

In this section we will prove:

Theorem 4.1. *Suppose M_j are precompact open metric spaces, (X, d_X) is a compact metric space and $(\bar{M}_j, d_{M_j}) \stackrel{GH}{\longrightarrow} (X, d_X)$ then for each $\delta > 0$, there exist a subsequence $\bar{M}^\delta_{j_k}$, a compact metric space $Y^\delta(j_k) \subset X$ such that*

$$\left(\bar{M}^\delta_{j_k}, d_{M_{j_k}}\right) \stackrel{GH}{\longrightarrow} \left(Y^\delta(j_k), d_\delta\right), \tag{68}$$

for any $\delta = \delta_i$ in the sequence and

$$Y^{\delta_1}(j_k) \subset Y^{\delta_2}(j_k) \tag{69}$$

if (68) hold for δ_1, δ_2, $0 < \delta_2 < \delta_1$.

Let

$$U_{\{\delta_i\},\{j_k\}} = \bigcup_{\delta_i} Y^{\delta_i}(j_k). \tag{70}$$

Then $U_{\{\delta_i\},\{j_k\}}$ is an open subset of X.

If we have two sequences $\{\delta_i\}, \{\beta_i\}$ such that (68) for all $\delta \in \{\delta_i\} \cup \{\beta_i\}$ then

$$U_{\{\delta_i\},\{j_k\}} = U_{\{\beta_i\},\{j_k\}}. \tag{71}$$

Note that M^δ_j can be an empty space. See Example 4.8. Consider the Gromov-Hausdorff limit of an empty metric space, to be an empty metric space.

Remark 4.2. *In Example 4.9 we see that a subsequence j_k may be necessary to obtain GH convergence of the δ inner regions and that $U_{\{\delta_i\},\{j_k\}}$ depends on the choice of the subsequence. In Example 4.11 we see that even the closure of $U_{\{\delta_i\},\{j_k\}}$ may depend on the choice of subsequence j_k. In Example 4.12 we see that $U_{\{\delta_i\},\{j_k\}}$ may be disjoint and not isometric.*

4.1. Hausdorff Convergence of δ Inner Regions.

We begin with a very basic theorem:

Theorem 4.3. *Let (Z, d_Z) be a compact metric space. Suppose $M_j \subset Z$ are open metric spaces with the induced metric and $X \subset Z$ is closed such that $\bar{M}_j \xrightarrow{H} X$ then for each $\delta > 0$, there exist a subsequence $\bar{M}^\delta_{j_k}$ and a compact set $W^\delta(j_k) \subset X$ such that*

$$\bar{M}^\delta_{j_k} \xrightarrow{H} W^\delta(j_k), \tag{72}$$

and if (72) holds for δ_1, δ_2, $0 < \delta_2 < \delta_1$, then

$$W^{\delta_1}(j_k) \subset W^{\delta_2}(j_k) \tag{73}$$

Given a sequence of positive numbers $\delta_i \to 0$ there exists a subsequence $\{j_k\} \subset \mathbb{N}$ such that (72) holds for all $\delta = \delta_i$. Let

$$U_{\{\delta_i\},\{j_k\}} = \bigcup_{\delta_i} W^{\delta_i}(j_k) \tag{74}$$

$U'_{\{\delta_i\},\{j_k\}}$ is an open subset of X. If we have two sequences $\{\delta_i\}, \{\beta_i\}$ such that (72) for all $\delta \in \{\delta_i\} \cup \{\beta_i\}$ then

$$U_{\{\delta_i\},\{j_k\}} = U_{\{\beta_i\},\{j_k\}}. \tag{75}$$

There are occasions where M^δ_j can be an empty space. We consider the Hausdorff limit of an empty metric space, to be an empty metric space.

Before we prove this theorem we provide an example demonstrating that even if

$$\bar{M}^{\delta_1}_{j_k} \xrightarrow{\mathrm{H}} W^{\delta_1}(j_k) \text{ and } \bar{M}^{\delta_2}_{j_k} \xrightarrow{\mathrm{H}} W^{\delta_2}(j_k), \tag{76}$$

for some $\delta_1 > \delta_2 > 0$, we cannot assure that for $\delta \in (\delta_2, \delta_1)$, $\bar{M}^\delta_{j_k}$ converges:

Example 4.4. *Fix $\varepsilon < 1/3$. In 2 dimensional Euclidean space, $\mathbb{E}^2$, consider the sequence M_j where M_{2j} is a ball of radius 1 with a spline of width 4ε attached to it as it is depicted in Figure 3, M_{2j+1} is a ball of radius 1 with a spline whose width decreases from 6ε to 4ε as $j \to \infty$. Then $\bar{M}^\varepsilon_j$ converges to ball of radius $1 - \varepsilon$ with a spline of width 2ε, $\bar{M}^{3\varepsilon}_j$ converges to a ball of radius $1 - 3\varepsilon$ with no spline attached. But $\bar{M}^{2\varepsilon}_{2j}$ converges to ball of radius $1 - 2\varepsilon$ while $\bar{M}^{2\varepsilon}_{2j+1}$ converges to ball of radius $1 - 2\varepsilon$ with a line segment attached to it. Thus $M^{2\varepsilon}_j$ does not converge in the Hausdorff sense.*

In the proof of Theorem 4.3 we will apply the following fact:

Remark 4.5. *Recall that if $\{A_j\}$ is a sequence of closed subsets of a metric space A such that $A_j \xrightarrow{H} A_\infty$, then*

$$A_\infty = \left\{ a \in A : \ \forall j \in \mathbb{N}\ \exists a_j \in A_j, \lim_{j\to\infty} a_j = a \right\}. \tag{77}$$

Any subsequence $\{A_{j_k}\}$ of $\{A_j\}$ also converges in Hausdorff sense to A_∞. Then

$$A_\infty = \left\{ a \in A : \ \forall k \in \mathbb{N}\ \exists a_{j_k} \in A_{j_k}, \lim_{k\to\infty} a_{j_k} = a \right\}. \tag{78}$$

We now prove Theorem 4.3:

Proof. Apply Theorem 2.16 to the sequence $\{\bar{M}^\delta_j\}_{j=1}^\infty$ to get a subsequence $\{\bar{M}^\delta_{j_k}\}_{k=1}^\infty$ and a compact set $W^\delta(j_k)$ such that (72) is satisfied. Since $\bar{M}^\delta_{j_k} \subset \bar{M}_{j_k}$ then $W^\delta(j_k) \subset X$. Similarly, $W^{\delta_1}(j_k) \subset W^{\delta_2}(j_k)$ when (72) holds for $0 < \delta_2 < \delta_1$.

Given $\delta_i \to 0$. Start with δ_1. By Theorem 2.16 there exists a subsequence $\{j_k(\delta_1)\}_{k=1}^{\infty}$ of $\{j\}_{j=1}^{\infty}$ and a compact set $W^{\delta_1}(j_k(\delta_1))$ such that $\bar{M}^{\delta_1}_{j_k(\delta_1)} \xrightarrow{\mathrm{H}} W^{\delta_1}(j_k(\delta_1))$. For $n > 1$, there exists a subsequence $\{j_k(\delta_n)\}_{k=1}^{\infty}$ of $\{j_k(\delta_{n-1})\}_{k=1}^{\infty}$ and a compact set $W^{\delta_n}(j_k(\delta_n))$ such that $\bar{M}^{\delta_n}_{j_k(\delta_n)} \xrightarrow{\mathrm{H}} W^{\delta_n}(j_k(\delta_n))$. Define $j_k = j_k(\delta_k)$. Then $\{j_k\}_{k=n}^{\infty}$ is a subsequence of $\{j_k(\delta_n)\}_{k=1}^{\infty}$ thus (72) holds for all n.

Let y be an element of $U^{'}_{\{\delta_i\},\{j_k\}}$. Then there exist $N \in \mathbb{N}$ such that $y \in W^{\delta_i}(j_k)$ for $i \geq N$. Suppose that $x \in X$ and $d_Z(x, y) < \delta_N/6$. Since $y \in W^{\delta_i}(j_k)$ choose $y_{j_k} \in \bar{M}^{\delta_N}_{j_k}$ such that $y = \lim_{j\to\infty} y_{j_k}$ and $d_Z(y, y_{j_k}) < \delta_N/6$. Analogously, take $x_j \in \bar{M}_j$ such that $x = \lim_{j\to\infty} x_j$ and $d_Z(x, x_j) < \delta_N/6$. Then

$$d_Z(x_{j_k}, y_{j_k}) < d_Z(x_{j_k}, x) + d_Z(x, y) + d_Z(y, y_{j_k}) < \delta_N/2. \tag{79}$$

This implies that $d_Z(x_{j_k}, \partial(M_{j_k})) > \delta_N/2$. Then $x \in W^{\delta_i}(j_k) \subset U^{'}_{\{\delta_i\},\{j_k\}}$ for some $i > N$.

If there is another sequence $\beta_i \to 0$ such that (72) holds for all $\delta = \beta_i$, for each i find l such that $\delta_l < \beta_i$ then $W^{\beta_i}(j_k) \subset W^{\delta_l}(j_k)$. This proves that $U^{'}_{\{\beta_i\},\{j_k\}} \subset U^{'}_{\{\delta_i\},\{j_k\}}$. The same reasoning works to prove

$$U^{'}_{\{\delta_i\},\{j_k\}} \subset U^{'}_{\{\beta_i\},\{j_k\}}$$

□

Definition 4.6. *With the hypothesis of Theorem 4.3 define*

$$U^{'}_{\{j_k\}} = \bigcup W^{\delta}(j_k) \tag{80}$$

where the union is taken over all δ for which $\bar{M}^{\delta}_{j_k}$ is a sequence that converges in Hausdorff sense to a metric space, $W^{\delta}(j_k)$, and

$$U^{'} = \bigcup_{\delta>0} W^{\delta} \tag{81}$$

where W^{δ} is the Hausdorff limit space of some convergent subsequence of $\bar{M}^{\delta}_j$.

4.2. **Finding Limits of Inner Regions in the Gromov-Haudorff limits.** In this subsection we prove Theorem 4.1:

Proof. By Theorem 2.15 there exists a common metric space Z and isommetric embeddings $\varphi_j : (\bar{M}_j, d_{M_j}) \to (Z, d_Z)$, $\varphi : (X, d_X) \to (Z, d_Z)$ such that $d_H^Z(\varphi_j(\bar{M}_j), \varphi(X)) \to 0$. Now we can apply Theorem 4.3. For each $\delta > 0$, there exist a subsequence $\varphi_{j_k}(\bar{M}^{\delta}_{j_k})$ and a compact set $W^{\delta}(j_k) \subset \varphi(X)$ such that $\varphi_{j_k}(\bar{M}^{\delta}_{j_k}) \xrightarrow{\mathrm{H}} W^{\delta}(j_k)$. Let $Y^{\delta}(j_k) = \varphi^{-1}(W^{\delta}(j_k))$. Clearly, (68) holds and $Y^{\delta_1}(j_k) \subset Y^{\delta_2}(j_k)$ when (68) hold for $0 < \delta_2 < \delta_1$. Given a sequence of positive numbers $\delta_i \to 0$ there exists a subsequence $\{j_k\} \subset \mathbb{N}$ such that φ_{j_k} $(\bar{M}^{\delta_i}_{j_k})$ $\xrightarrow{\mathrm{H}} W^{\delta_i}(j_k)$ for all i, then (68) hold for all i and $U_{\{\delta_i\},\{j_k\}} = \varphi^{-1}(U^{'}_{\{\delta_i\},\{j_k\}})$ is an open subset of X that does not depend on the sequence δ_i. □

4.3. **Unions of the Limits of Inner Regions in the Gromov-Haudorff limits.**

Definition 4.7. *With the hypothesis of Theorem 4.1 define*

$$U_{\{j_k\}} = \bigcup Y^{\delta}(j_k) \tag{82}$$

where the union is taken over all δ such that $\bar{M}^{\delta}_{j_k}$ is a sequence that converges in Gromov-Hausdorff sense to $Y^{\delta}(j_k)$. Let

$$U = \bigcup_{\delta>0} Y^{\delta} \tag{83}$$

where Y^{δ} is the Gromov-Hausdorff limit space of some convergent subsequence of $\bar{M}^{\delta}_j$.

In Remark 6.7, we will explain that the regions U defined in Definition 4.7 are special cases of the glued limits we will construct in Theorem 6.3. Since they are easy to understand in this setting we present a few examples here and more later.

Example 4.8. *Let M_j be a Euclidean disk of radius $1/j$. Then $\bar{M}_j \xrightarrow{GH} X$ where X is a single point. For any $\delta > 0$, taking $j > 1/\delta$, we see that M^{δ}_j are empty spaces. Thus U is the empty set.*

In the following example we see that $U_{\{j_k\}}$ depends on the subsequence $\{j_k\}$ and in Example 4.10, we see that X is not necessarily contained in the closure of U.

Example 4.9. *Let M_{2j} be the standard Euclidean disk of radius 1 and let M_{2j+1} be the standard Euclidean disk with the center point removed. Then $\bar{M}_j$ is a closed Euclidean disk as is the limit space X. Given $\delta \in (0,1)$, M^{δ}_{2j} is the standard Euclidean disk of radius $1-\delta$. Their metric completions converge to the closed disk of radius $1-\delta$. $U_{\{2j\}}$ is the open Euclidean disk of radius 1. However M^{δ}_{2j+1} is a Euclidean annulus, $Ann_0(\delta, 1-\delta)$, and the metric completions converge to the closure of this annulus. $U_{\{2j+1\}}$ is the open Euclidean disk of radius 1 with the center point removed. In this example $U = U_{\{2j\}}$.*

Example 4.10. *In 2 dimensional euclidean space consider the sequence of a ball with a spline attached to it as it is depicted in Figure 3. The Gromov-Hausdorff limit of the sequence is a ball with an interval attached while the closure of U is just the closed ball.*

In Example 4.9, we saw $U_{\{2j\}} \neq U_{\{2j+1\}}$ still their closures are the same. This is not always the case. $U_{\{j_k\}}$ could even be an empty set.

Example 4.11. *For $j \in \mathbb{N}$, let M_{2j} be a flat torus so it has no boundary and M_{2j+1} be flat tori with increasingly dense small holes cut out but the holes get smaller and smaller so that they still converge to the flat torus X. $U_{\{2j\}} = X$ but for any $\delta > 0$, M^{δ}_{2j+1} becomes an empty set. So $U_{\{2j+1\}}$ is the empty set.*

Example 4.12. *For $j \in \mathbb{N}$, let M_{2j} be a flat torus $S^1 \times S^1$, with increasingly many dense small holes in $W \times S^1$, where $W = (0, \pi/4) \subset S^1$ and let M_{2j+1} be a flat torus $S^1 \times S^1$, with increasingly many dense small holes in $(S^1 \setminus W) \times S^1$. Then*

$$U_{\{2j\}} = (S^1 \setminus W) \times S^1 \quad and \quad U_{\{2j+1\}} = W \times S^1 \tag{84}$$

with the restricted distance from $S^1 \times S^1$ which are disjoint and not isometric to each other.

5. Converging Inner Regions of Sequences with Curvature Bounds

In this section we prove δ inner regions converge under certain geometric hypothesis on the manifolds even when the manifolds themselves have no Gromov-Hausdorff limits.

5.1. **Constant Sectional Curvature.** Here we prove that the inner regions of a sequence of manifolds in the following class have a subsequence which converges in the Gromov-Hausdorff sense.

Definition 5.1. *Given $m \in \mathbb{N}$, $H \in \mathbb{R}$, $V > 0$, and $l > 0$ we define $\mathcal{M}_H^{m,V,l}$ to be the class of connected open Riemannian manifolds, M, of dimension $\leq m$, with constant sectional curvature $Sect_M = H$, $\mathrm{Vol}(M) \leq V$, and*

$$L_{min}(M) = \inf\{L_g(C) : \ C \textit{ is a closed geodesic in } M\} > l. \tag{85}$$

where a closed geodesic is any geodesic which starts and ends at the same point.

Recall that complete simply connected manifolds with constant sectional curvature $H \leq 0$ have no closed geodesics by the Hadamard Theorem while those with $H > 0$ have $L(M) = 2\pi/\sqrt{H}$ (c.f. [5]). Here we are requiring that the closed geodesic lies in an open manifold M and we do not have completeness.

Theorem 5.2. *Given any $\delta > 0$ if $(M_j, g_j) \subset \mathcal{M}_H^{m,V,l}$, then there is a subsequence $(M_{j_k}^\delta, d_{M_{j_k}})$ such that the metric completion with the restricted metric converges in the Gromov-Hausdorff sense to a metric space (Y^δ, d). In particular the extrinsic diameters measured using the restricted metric are bounded uniformly*

$$\mathrm{Diam}(M_{j_k}^\delta, d_{M_{j_k}}) \leq \epsilon_0 \frac{V}{V_H^m(\epsilon_0)} \tag{86}$$

$$\mathrm{Diam}(Y^\delta, d) \leq \epsilon_0 \frac{V}{V_H^m(\epsilon_0)}. \tag{87}$$

where

$$\epsilon_0 = min\{\delta, l/2, \pi/\sqrt{H}\}/2 \textit{ if } H > 0 \tag{88}$$

$$\epsilon_0 = min\{\delta, l/2\}/2 \textit{ otherwise}, \tag{89}$$

and $V_H^m(\epsilon_0)$ is the volume of a ball of radius ϵ_0 in the complete simply connected space with constant sectional curvature H.

Remark 5.3. *There are no closed geodesics in the M_j of Examples 1.1 and 1.2, so $L(M_j) = \infty$. These examples have $H = 0$ and $m = 2$. Since Example 1.2 also has a uniform upper bound on volume, it demonstrates why we can only obtain Gromov-Hausdorff convergence of the M_j^δ instead of the M_j themselves. The M_j^δ of Example 1.1 do not have Gromov-Hausdorff converging subsequences (see Remark 5.5) demonstrating the necessity of the hypothesis requiring an upper volume bound.*

Proof. Let $M \in \mathcal{M}_H^{m,V,l}$ and $p \in M^\delta$. Recall that $\epsilon_0 = min\{\delta, l/2, \pi/\sqrt{H}\}/2$ if $H > 0$, $\epsilon_0 = min\{\delta, l/2\}/2$ otherwise. Then for $0 < \epsilon < \epsilon_0$, $B_p(\epsilon)$ does not reach the boundary of M and does not contain any conjugate point to p since one does not reach a conjugate point before one would in the comparison space.

We claim that there are also no cut points to p in $B_p(\epsilon)$. If there was a cut point, q, then proceeding as in a similar way to Klingenberg [8], there exists a closed geodesic starting at p of length $\leq 2d(p,q) < 2\epsilon_0$. By hypothesis, the length of this closed geodesic is greater than l which is a contradiction.

Thus there is a Riemannian isometric diffeomorphism

$$\psi : B_p(\epsilon_0) \to B_x(\epsilon_0) \subset M_H^m \tag{90}$$

where M^m_H is the simply connected space of constant sectional curvature, H. In particular $\mathrm{Vol}(B_p(\epsilon))$ is greater or equal than the volume of a ball of the same radius in a simply connected space form of constant curvature H. By combining Proposition 2.11 with Proposition 2.10 and then Gromov's Compactness Theorem there is a subsequence $(M^\delta_{j_k}, d_{M_{j_k}})$ such that the metric completion with the restricted metric converges in the Gromov-Hausdorff sense to a metric space (Y^δ, d). Notice that by Proposition 2.11, the maximum number of disjoint balls of radius $\epsilon_0/2$ that lie in M is $\leq \frac{V}{V^m_H}(\epsilon_0/2)$. Thus, by Proposition 2.10, the minimum number of balls of radius ϵ_0 to cover M is $\leq \frac{V}{V^m_H}(\epsilon_0/2)$. From this follows that

$$\mathrm{Diam}(M^\delta_{j_k}, d_{M_{j_k}}) \leq \epsilon_0 \frac{V}{V^m_H(\epsilon_0/2)}. \tag{91}$$

Since

$$\mathrm{Diam}(M^\delta_{j_k}, d_{M_{j_k}}) \to \mathrm{Diam}(Y^\delta, d), \tag{92}$$

we conclude that

$$\mathrm{Diam}(Y^\delta, d) \leq \epsilon_0 \frac{V}{V^m_H(\epsilon_0/2)}. \tag{93}$$

□

Remark 5.4. *If the injectivity radius for each $p \in M^\delta_j$ is bounded above by a positive constant then the condition in the lenght of closed geodesics in Theorem 5.2 is satisfied.*

5.2. **Examples with Constant Sectional Curvature.** The volume condition in Theorem 5.2 may not be replaced by a condition on diameter:

Remark 5.5. *Let (M_j, g_j) be the j^{th} covering space of $Ann_{(0,0)}(1/j, 1) \subset \mathbb{E}^2$.*

Since every point in M_j is less than a distance 1 from the inner boundary, and the inner boundary has length $j2\pi(1/j) = 2\pi$, we know

$$\mathrm{Diam}(M_j, d_{M_j}) \leq 2\pi + 2. \tag{94}$$

Yet the number of disjoint balls of radius $\delta < 1/4$ centered on the cover of $\partial B_{(0,0)}(1/2)$ is greater than $2j$. So there is no subsequence of M^δ_j which converges in the Gromov-Hausdorff sense.

This sequence fails to satisfy the volume condition of Theorem 5.2:

$$\mathrm{Vol}(M_j) = j(\pi 1^2 - \pi/j^2) = \pi(j^2 - 1)/j. \tag{95}$$

It is worth observing that the intrinsic diameters

$$\mathrm{Diam}(M^\delta_j, M^\delta_j) \geq j2\pi(\delta + 1/j) \tag{96}$$

also diverges to infinity.

Remark 5.6. *The flat manifolds of Example 1.2 described explicitly in Example 2.13 satisfies the hypothesis of Theorem 5.2. See Figure 5. In fact for fixed $\delta > 0$, once $(2\pi/j)4 < \delta$, every point with $r \geq 2$ lies within a distance δ from the boundary because the spline is less than δ wide. So all the M^δ_j eventually lie within $r < 2$, where the metric is just the standard Euclidean metric and there is a uniform bound on the number of disjoint balls. So the Gromov-Hausdorff limit also lies within the Euclidean ball of radius 2. On the other hand, every point within the ball of radius $1+\delta < r < 2-\delta$, lies in M^δ_j, so the Gromov-Hausdorff limit Y^δ contains $Ann_{(0,0)}(1+\delta, 2-\delta)$. In fact Y^δ is the metric completion of this annulus with the flat Euclidean metric.*

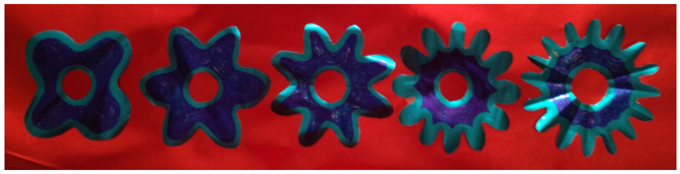

FIGURE 5. Models of Example 1.2: M^δ_4, M^δ_6, M^δ_8, M^δ_{12}, M^δ_{16}...

5.3. **Manifolds with Nonnegative Ricci Curvature.** Here we prove Theorem 1.4 by applying Gromov's Compactness Theorem (c.f. Theorem 2.9) combined with the following proposition:

Proposition 5.7. *If (M, g_M) is a compact Riemannian manifold with boundary that has nonnegative Ricci curvature, then for any $\delta > 0$, and any $\epsilon \in (0, \delta/2)$, the δ inner region, M^δ, contains a finite collection of points $\{p_1, p_2, ...p_N\}$, such that*

$$M^\delta \subset \bigcup_{i=1}^N B_{p_i}(\epsilon) \tag{97}$$

where

$$N \le N(\delta, \epsilon, D_\delta, V, \theta) = \frac{V}{\theta}\left(\frac{2^{2D_\delta/\epsilon}}{\epsilon}\right)^m \tag{98}$$

where $m = \dim(M)$, $\mathrm{Vol}(M) \le V$,

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) \le D_\delta, \tag{99}$$

and

$$\sup\{\mathrm{Vol}(B_q(\delta)) : \ q \in M^\delta\} \ge \theta\delta^m \tag{100}$$

Remark 5.8. *Note that in this proposition, we can use the volume of any ball centered in M^δ to estimate θ in (100). This allows us to study sequences like those in Example 3.1. One does not need a Ricci curvature condition if one has a uniform lower bound on the volumes of all balls centered in M^δ as can be seen in Proposition 2.11 in the review of Gromov-Hausdorff convergence.*

Proof. By Propositions 2.10 and 2.11 in the review of Gromov-Hausdorff convergence, we need only TO find a uniform lower bound on the volume of an arbitrary ball $B_p(\epsilon)$ centered at $p \in M^\delta$.

Fix q achieving the supremum in (200). Then by the fact that $B_q(\delta)$ does not hit ∂M and M has nonnegative Ricci curvature, we may apply the Bishop-Gromov Volume Comparison Theorem to see that,

$$\theta \le \frac{\mathrm{Vol}(B_q(\delta))}{\delta^m} \le \frac{\mathrm{Vol}(B_q(\epsilon))}{\epsilon^m}, \tag{101}$$

because $\delta > \delta/2 > \epsilon$.

Let $C : [0, 1] \to M^\delta$ be the shortest curve from p to q. Then

$$L = L(C) \le \mathrm{Diam}(M^\delta, d_{M^\delta}) \le D_\delta. \tag{102}$$

Let $n > L/\epsilon$ and $x_j = C(t_j)$ where $t_j = jL/n$, so that

$$x_j \in M^\delta \text{ and } d_M(x_{j-1}, x_j) = L/n < \epsilon. \tag{103}$$

In particular $B_{x_j}(2\epsilon)$ lies within the interior of M and has nonnegative Ricci curvature. Thus by the Bishop-Gromov Volume Comparison Theorem,

$$\mathrm{Vol}(B_{x_j}(\epsilon)) \geq \frac{1}{2^m}\mathrm{Vol}(B_{x_j}(2\epsilon)) \tag{104}$$

$$\geq \frac{1}{2^m}\mathrm{Vol}(B_{x_{j+1}}(\epsilon)). \tag{105}$$

Applying this repeatedly from $j = 1$ to n, and finally applying (101), we have

$$\mathrm{Vol}(B_p(\epsilon)) \geq \frac{1}{2^{mn}}\mathrm{Vol}(B_q(\epsilon)) \tag{106}$$

$$\geq \frac{1}{2^{mD_\delta/\epsilon}}\mathrm{Vol}(B_q(\epsilon)) \tag{107}$$

$$\geq \frac{1}{2^{mD_\delta/\epsilon}}\theta\epsilon^m \tag{108}$$

The estimate on $N(\delta, \epsilon, D_\delta, V, \theta)$ then follows immediately from Propositions 2.10 and 2.11. □

6. Glued Limit Spaces

In this section we define glued limit spaces and study their properties without any assumptions on curvature. We begin by constructing isometric embeddings $\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_{i+1}} \to Y^{\delta_i}$ between the Gromov Hausdorff limits, Y^{δ_i}, of inner regions, $M_j^{\delta_i}$ [Theorem 6.1]. We then apply these isometric embeddings to glue together, Y^{δ_i}, and construct a glued limit space, $Y = Y(\{\delta_i\}, \{\varphi_{\delta_{i+1},\delta_i}\})$ [Theorem 6.3]. These glued limit spaces depend on the isometric embeddings as seen in Example 6.16.

The glued limit space is unique when the M_j converge in the Gromov-Hausdorff sense [Remark 6.7]. In Remark 6.10 we describe how Example 2.13, which had no Gromov-Haudorff limit, has a bounded and precompact glued limit space. We provide another example with a bounded glued limit space which is not precompact [Example 6.12]. We provide an example where the glued limit space is not a length space [Remark 6.13].

6.1. Gluing the Inner Regions Together.

Here we prove the existence of isometric embeddings which we will later apply as glue to connect the inner regions together.

Theorem 6.1. *Suppose that $\delta_i \to 0$ is decreasing, $i = 0, 1, 2, \cdots$, and one has a sequence of open manifolds M_j such that*

$$(\bar{M}_j^{\delta_i}, d_{M_j}) \xrightarrow{GH} (Y^{\delta_i}, d_{Y^{\delta_i}}) \tag{109}$$

for all i, where possibly some of these sequences and their limits are eventually empty sets. Then there exist isometric embeddings:

$$\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_i} \to Y^{\delta_{i+1}}, \tag{110}$$

which are just the identity when $\delta_i = \delta_{i+1}$. If $\delta \in (0, \delta_0]$ there exists a compact metric space, $Y^\delta \subset Y^{\delta_{i+1}}$ with the restricted metric $d_{Y^\delta} = d_{Y^{\delta_{i+1}}}$, and a converging subsequence

$$(\bar{M}_{j_k}^{\delta}, d_{M_{j_k}}) \xrightarrow{GH} (Y^\delta, d_{Y^\delta}) \tag{111}$$

and when $\delta \in (\delta_{i+1}, \delta_i)$ for any such Y^δ the restriction map, $\varphi_{\delta,\delta_i} : Y^{\delta_i} \to Y^\delta$, and the inclusion map, $\varphi_{\delta_{i+1},\delta} : Y^\delta \to Y^{\delta_{i+1}}$ are isometric embeddings.

Proof. By Theorem 2.15 there exists a compact metric space Z and isometric embeddings

$$\varphi_j : \bar{M}_j^{\delta_{i+1}} \to Z \text{ and } \varphi_\infty : Y^{\delta_{i+1}} \to Z \tag{112}$$

such that

$$\varphi_j(\bar{M}_j^{\delta_{i+1}}) \xrightarrow{\text{H}} \varphi_\infty(Y^{\delta_{i+1}}). \tag{113}$$

By Theorem 2.16 we can choose a subsequence $\{j_k\}_{k=1}^\infty$ such that $\varphi_{j_k}(\bar{M}_{j_k}^{\delta_i})$ converge in Hausdorff sense to a compact subspace $X^{\delta_i} \subset \varphi_\infty(Y^{\delta_{i+1}})$. By the hypothesis,

$$\bar{M}_{j_k}^{\delta_i} \xrightarrow{\text{GH}} Y^{\delta_i}. \tag{114}$$

Then by uniqueness up to an isometry of the Gromov-Hausdorff limit space there exists an isometric embedding

$$\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_i} \to Y^{\delta_{i+1}} \text{ such that } \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i}) = \varphi_\infty^{-1}(X^{\delta_i}). \tag{115}$$

By Theorem 2.12 there is a uniform upper bound, $D_i > 0$, of the diameters of $(\bar{M}_j^{\delta_i}, d_{M_j})$ and a function $N_i : (0, D_i] \to \mathbb{N}$ such that $N_i(\epsilon)$ is an upper bound for the number of $\epsilon - balls$ needed to cover $\bar{M}_j^{\delta_i}$ for all $\epsilon \in (0, D_i]$ and for all $j \in \mathbb{N}$. If $\delta \in (\delta_{i+1}, \delta_i)$, define $N : (0, D_{i+1}/2] \to \mathbb{N}$ by $N(\epsilon) = N_{i+1}(2\epsilon)$. Then

$$\operatorname{Diam}(\bar{M}_j^{\delta}, d_{M_j}) \le \operatorname{Diam}(\bar{M}_j^{\delta_{i+1}}, d_{M_j}) \le D_{i+1}. \tag{116}$$

Apply Theorem 2.9 to get a subsequence $\{l_k\}_{k=1}^\infty$ of $\{j_k\}_{k=1}^\infty$ such that $\varphi_{l_k}(\bar{M}_{l_k}^{\delta_i})$ converge in Hausdorff sense to a closed subset $X^\delta \subset \varphi_\infty(Y^{\delta_{i+1}})$.

We define

$$Y^\delta = \varphi_\infty^{-1}(X_\delta) \subset Y^{\delta_{i+1}}. \tag{117}$$

The choice of a subsequence $\{l_k\}_{k=1}^\infty$ of $\{j_k\}_{k=1}^\infty$ implies that $X^{\delta_i} \subset Y^\delta$. So $Y^{\delta_i} \subset Y^\delta$. The rest of the theorem immediately follows. □

Remark 6.2. *The choice of isometric embeddings $\varphi_{\delta_{i+1},\delta_i}$ is not unique. See Example 6.16 where we provide two distinct isometric embeddings $\varphi_{\delta_{i+1},\delta_i} \neq \varphi'_{\delta_{i+1},\delta_i}$.*

6.2. **Glued Limit Spaces are Defined.**

We can now define a glued limit space, depending on a sequence $\delta_i \to 0$, prove it is a metric space and prove that it contains isometric images of all Gromov-Haudorff limits of converging subsequences of inner regions.

Theorem 6.3. *Under the hypothesis of Theorem 6.1 one can define a glued limit space*

$$Y = Y(\{\delta_i\}, \{\varphi_{\delta_{i+1},\delta_i}\}) = Y^{\delta_0} \sqcup \left(\sqcup_{i=1}^\infty \left(Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}\left(Y^{\delta_i}\right)\right)\right) \tag{118}$$

with the metric:

$$d_Y(x,y) = \begin{cases} d_{Y^{\delta_0}}(x,y) & \text{if } x,y \in Y^{\delta_0}, \\ d_{Y^{\delta_{i+1}}}(x,y) & \text{if } x,y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}\left(Y^{\delta_i}\right), \\ d_{Y^{\delta_{i+1}}}\left(x, \varphi_{\delta_{i+1},\delta_0}(y)\right) & \text{if } x \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}\left(Y^{\delta_i}\right) \\ & \text{for some } i \in \mathbb{N} \text{ and } y \in Y^{\delta_0}, \\ d_{Y^{\delta_{i+j+1}}}\left(x, \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)\right) & \text{if } x \in Y^{\delta_{i+j+1}} \setminus \varphi_{\delta_{i+j+1},\delta_{i+j}}\left(Y^{\delta_{i+j}}\right) \\ & \text{and } y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}\left(Y^{\delta_i}\right) \text{ for some } i,j \in \mathbb{N} \end{cases}$$

where we set

$$\varphi_{\delta_{i+j},\delta_i} = \varphi_{\delta_{i+j},\delta_{i+j-1}} \circ \cdots \circ \varphi_{\delta_{i+1},\delta_i}. \tag{119}$$

For all $\delta \in (0,\delta_0]$ there exists a subsequence $M^\delta_{j_k}$ which converges in Gromov-Hausdorf sense to a compact metric space Y^δ and for any such Y^δ there exists an isometric embedding

$$F_\delta = F_{\delta,\{\delta_i\}} : Y^\delta \to Y. \tag{120}$$

such that for the δ_i in our sequence we have

$$F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^\delta_{i+1}) \tag{121}$$

If β_j is any sequence decreasing to 0, then

$$Y = \bigcup_{i=1}^{\infty} F_{\beta_j}(Y^{\beta_j}). \tag{122}$$

Remark 6.4. *In Example 4.10 for sufficiently large δ_0 each limit Y^δ is a ball in Euclidean 2-dimensional space. According to Definition 6.3 the glued limit space of this sequence is constructed by taking the disjoint union of the ball Y^{δ_0} with concentric annulus $Y^{\delta_0/i+1} \setminus Y^{\delta_0/i}$.*

Remark 6.5. *Note that the definition of the glued limit space depends on the choice of isometric embeddings in Theorem 6.1. See Example 6.16.*

Proof. We first prove that d_Y is positive definite. For the first and second cases of the definition of d_Y, we immediately see that $d_Y(x,y) = 0$ iff $x = y$. For the third and fourth cases, notice that $\varphi_{\delta_{i+j+1},\delta_{i+1}}(y) = (\varphi_{\delta_{i+j+1},\delta_{i+j}} \circ \varphi_{\delta_{i+j},\delta_{i+1}})(y)$ then

$$\varphi_{\delta_{i+j+1},\delta_{i+1}}(y) \in \varphi_{\delta_{i+j+1},\delta_{i+j}}(Y^{\delta_{i+j}}). \tag{123}$$

Thus $x \neq \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)$ and $d_Y(x,y) = d_{Y^{\delta_{i+j+1}}}\left(x, \varphi_{\delta_{i+j+1},\delta_{i+1}}(y)\right) \neq 0$.

Define $F_{\delta_i} : Y^{\delta_i} \to Y$ in the following way:

$$F_{\delta_i}(y) = \begin{cases} y & i = 1 \\ y & i > 1, \quad y \in Y^{\delta_i} \setminus \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}}) \\ \varphi^{-1}_{\delta_i,\delta_0}(y) & i > 1, \quad \varphi^{-1}_{\delta_i,\delta_0}(y) \in Y^{\delta_0} \\ \varphi^{-1}_{\delta_i,\delta_j}(y) & i > 1, \quad \varphi^{-1}_{\delta_i,\delta_j}(y) \in Y^{\delta_j} \setminus \varphi_{\delta_j,\delta_{j-1}}(Y^{\delta_{j-1}}) \\ & \text{for some } j > 1 \end{cases}$$

What we are doing in the third and fourth part of the definition of F_{δ_i} is the following. Suppose that $y \in Y^{\delta_0/i}$ then either

$$y \in Y^{\delta_i} \setminus \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}}) \tag{124}$$

or $y \in \varphi_{\delta_i,\delta_{i-1}}(Y^{\delta_{i-1}})$. In the latter case there exists $y_{i-1} \in Y^{\delta_{i-1}}$ such that $y = \varphi_{\delta_i,\delta_{i-1}}(y_{i-1})$. If $i-1 > 1$, either $y_{i-1} \in Y^{\delta_{i-1}} \setminus \varphi_{\delta_{i-1},\delta_{i-2}}(Y^{\delta_{i-2}})$ or $y_{i-1} \in \varphi_{\delta_{i-1},\delta_{i-2}}(Y^{\delta_{i-2}})$. Proceeding in the same way, if necessary, we find j such that there exists $y_j \in Y^{\delta_j} \setminus \varphi_{\delta_j,\delta_{j-1}}(Y^{\delta_{j-1}})$ when $j > 1$ and $y_j \in Y^{\delta_j}$ when $j = 1$, that satisfies $y = \varphi_{\delta_i,\delta_j}(y_j)$.

It is easy to see that

$$F_{\delta_i}(Y^{\delta_i}) = Y^{\delta_0} \cup \left(\sqcup_{j=1}^{i-1} \left(Y^{\delta_{j+1}} \setminus \varphi_{\delta_{j+1},\delta_j}\left(Y^{\delta_j}\right)\right)\right), \tag{125}$$

and for $j < i$,

$$F_{\delta_j} = F_{\delta_i} \circ \varphi_{\delta_i,\delta_j} \tag{126}$$

For arbitrary δ, by Theorem 6.1, there exists a subsequence, $M^{\delta}_{j_k}$, which converges in the GH sense to a limit Y^{δ}. Let $F_\delta : Y^\delta \to Y$

$$F_\delta = F_{\{\delta,\{\delta_i\}\}} = \begin{cases} F_{\delta_0} \circ \varphi_{\delta_0,\delta} & \delta_0 < \delta \\ F_{\delta_{i+1}} \circ \varphi_{\delta_{i+1},\delta} & \delta_{i+1} \le \delta < \delta_i \end{cases}$$

where $\varphi_{\delta_0,\delta}, \varphi_{\delta_{i+1},\delta}$ are given in Theorem 6.1.

Observe that in the latter case of the definition of F_δ, $F_{\delta_i} = F_\delta \circ \varphi_{\delta,\delta_i}$. This and the definition of F_δ gives:

$$F_{\delta_i}(Y^{\delta_i}) \subset F_\delta(Y^\delta) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \tag{127}$$

Now we have β_j decreasing to 0, there exists N sufficiently large that $\beta_j \le \delta_0$, and for all $j \ge N$ we have $\exists i_j$ such that $\beta_j \in [\delta_{i+1}, \delta_i)$. From (125) and (127), taking $\delta = \beta_i$, we conclude that

$$Y = \bigcup_{j=N}^{\infty} F_{\beta_j}(Y^{\beta_j}) = \bigcup_{j=1}^{\infty} F_{\beta_j}(Y^{\beta_j}). \tag{128}$$

because $F_{\beta_0}(Y^{\beta_0}) \subset F_{\beta_N}(Y^{\beta_N})$.

To prove that F_δ is an isometric embedding it is enough to prove that for each F_{δ_i}. F_{δ_0} is an isometric embedding by definition of Y. For $F_{\delta_{i+1}}$ we must check three cases. Let $x, y \in Y^{\delta_{i+1}}$. First case: $x, y \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$ then $F_{\delta_{i+1}}(x) = x$ and $F_{\delta_{i+1}}(y) = y$. Second case: $x \in Y^{\delta_{i+1}} \setminus \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$ and $y \in \varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})$. Thus, $F_{\delta_{i+1}}(y) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-j}}(y) \in Y^{\delta_{i+1-j}} \setminus \varphi_{\delta_{i+1-j},\delta_{i-j}}(Y^{\delta_{i-j}})$ for some j. So

$$\begin{aligned} d_Y(F_{\delta_{i+1}}(x), F_{\delta_{i+1}}(y)) &= d_{Y^{\delta_{i+1-j}}}(F_{\delta_{i+1}}(x), \varphi_{\delta_{i+1},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))) \\ &= d_{Y^{\delta_{i+1}}}(x, y). \end{aligned}$$

Finally, if $F_{\delta_{i+1}}(x) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-k}}(x)$, $F_{\delta_{i+1}}(y) = \varphi^{-1}_{\delta_{i+1},\delta_{i+1-j}}(y)$. Suppose that $k \le j$. Recall that $\varphi_{\delta_{i+1},\delta_{i+1-k}} \circ \varphi_{\delta_{i+1-k},\delta_{i+1-j}} = \varphi_{\delta_{i+1},\delta_{i+1-j}}$, then

$$\begin{aligned} d_Y(F_{\delta_{i+1}}(x), F_{\delta_{i+1}}(y)) &= d_{Y^{\delta_{i+1-k}}}(F_{\delta_{i+1}}(x), \varphi_{\delta_{i+1-k},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))) \\ &= d_{Y^{\delta_{i+1}}}(\varphi_{\delta_{i+1},\delta_{i+1-k}}(F_{\delta_{i+1}}(x)), \varphi_{\delta_{i+1},\delta_{i+1-j}}(F_{\delta_{i+1}}(y))) \\ &= d_{Y^{\delta_{i+1}}}(x, y). \end{aligned}$$

The triangle inequality follows from the above paragraphs. For $x, y, z \in Y$, find δ such that $x, y, z \in F_\delta(Y^\delta)$. The triangle inequality holds for the preimages of x, y, z and since F_δ is an isometric embedding, it also holds for x, y, z. □

6.3. **Glued Limits within Gromov-Hausdorff Limits.** In this section we prove that if M_j have a Gromov Hausdorff limit, then the glued limits of Gromov-Hausdorff converging sequences are the unions of the limits of the inner regions explored in Theorem 4.1 [Remark 6.7, Lemma 6.9]. As a consequence Examples 4.9- 4.12 are examples of glued limit spaces.

Remark 6.6. *Suppose M_j are Riemannian manifolds with boundary and $M_j \xrightarrow{GH} X$. In Example 8.1, we will see it is possible that M_j satisfy the conditions of Theorem 6.1 only trivially. In fact, it is possible that for each $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that $M^{\delta}_j = \emptyset$ for all $j \ge N_\delta$.*

Remark 6.7. *Assume $M_j \xrightarrow{GH} X$, and δ_i is a decreasing sequence, then by Theorem 4.1, for each δ_i there is a subsequence of δ_i inner regions, $M_j^{\delta_i}$, which GH converge to $Y^{\delta_i}(j_k) \subset X$. We can diagonalize to find a subsequence M_{j_k} such that for all i*

$$M_{j_k}^{\delta_i} \xrightarrow{GH} Y^{\delta_i}(j_k) \subset X \text{ as } k \to \infty. \tag{129}$$

Thus we have the hypothesis of Theorems 6.1 and 6.3, and so M_{j_k} has a glued limit space $Y = Y(\{\delta_i\}, \{\varphi_{\delta_{i+1},\delta_i}\})$. On the other hand by Theorem 4.1 and Definition 4.7 applied to this diagonalized subsequence M_{j_k}, we have

$$U = U_{\{\delta_i\},\{j_k\}} = \bigcup Y^{\delta_i} \subset X. \tag{130}$$

This U is in fact the glued limit space, Y, by Lemma 6.9 below. Thus the glued limit space does not depend on $\{\delta_i\}$ as long as we have the same subsequence j_k.

Remark 6.8. *Note that the glued limit may be a proper subset of X as seen in Example 4.10 whose glued limit was a disk in Euclidean space while X is a disk with a line segment attached.*

Lemma 6.9. *The gluing embeddings, $\varphi_{\delta_{i+1},\delta_i}$, of Theorem 4.1 are the inclusion maps*

$$Y^{\delta_i} \subset Y^{\delta_{i+1}} \subset U \subset X, \tag{131}$$

Proof. By Theorem 2.15 there exist a common metric space Z and isommetric embeddings $\varphi_j : (\bar{M}_j, d_{M_j}) \to (Z, d_Z)$, $\varphi : (X, d_X) \to (Z, d_Z)$ such that $d_H^Z(\varphi_j(\bar{M}_j), \varphi(X)) \to 0$. Applying Theorem 4.3 there exists a subsequence $\{j_k\} \subset \mathbb{N}$ such that $\varphi_{j_k}(\bar{M}_{j_k}^{\delta_i}) \xrightarrow{\mathrm{H}} W^{\delta_i}(j_k)$ for all i. Thus $\bar{M}_{j_k}^{\delta_i} \xrightarrow{\mathrm{GH}} \varphi^{-1}(W^{\delta_i}(j_k))$ for all i. If $\delta \in (\delta_i, \delta_{i+1})$ get a subsequence such that $\bar{M}_{j'_k}^{\delta} \xrightarrow{\mathrm{GH}} \varphi^{-1}(W^{\delta}(j'_k))$

By uniqueness up to an isometry of the Gromov-Hausdorff limit space for all δ there exist isometries $\phi_\delta : \varphi^{-1}(W^\delta(j_k)) \to Y^\delta$. Define

$$\varphi_{\delta_{i+1},\delta_i} = \phi_{\delta_{i+1}} \circ \phi_{\delta_i}^{-1}. \tag{132}$$

If $\delta \in (\delta_i, \delta_{i+1})$ then

$$\varphi_{\delta,\delta_i} = \phi_\delta \circ \phi_{\delta_i}^{-1} \tag{133}$$

and

$$\varphi_{\delta_{i+1},\delta} = \phi_{\delta_{i+1}} \circ \phi_\delta^{-1}. \tag{134}$$

□

6.4. **Glued Limit Spaces when there are no Gromov-Hausdorff limits.** In the setting of Theorem 6.1, the subsequence of manifolds M_j such that $M_j^\delta \xrightarrow{\mathrm{GH}} Y^\delta$ need not have any Gromov-Hausdorff limit. Here we discuss an old example and present two new examples.

Remark 6.10. *In Example 2.13 which had increasingly many splines, it was seen that the Gromov-Hausdorff limit for the sequence M_j described there does not exist. However the sequence M_j^δ converges to the metric completition of the annulus $Ann_{(0,0)}(1+\delta, 2-\delta)$ with the flat metric, see Remark 5.6. Start with $\delta_0 < 1/2$ then*

$$Y^{\delta_0} = Ann_{(0,0)}[1+\delta_0, 2-\delta_0], \tag{135}$$

and

$$Y^{\delta_0/(i+1)} \setminus \varphi_{\delta_0/(i+1),\delta_0/i}\left(Y^{\delta_0/i}\right) = A_1 \cup A_2 \tag{136}$$

where

$$A_1 = Ann_{(0,0)}[1+\delta_0/(i+1), 1+\delta_0/i\,) \text{ and} \tag{137}$$

$$A_2 = Ann_{(0,0)}(2-\delta_0/i, 2-\delta_0/(i+1)]. \tag{138}$$

Thus $Y = Ann_{(0,0)}(1,2)$ with the flat length metric. This glued limit space Y is precompact.

A similar example, also constructed using flat $M_j \subset \mathbb{E}^2$ with no Gromov-Haudorff limit has converging M_j^δ, and a glued limit space which is a flat open manifold that is bounded but not precompact:

Example 6.11. *We define a flat open manifold with j splines of decreasing width:*

$$M_j = U_j \cup V_j \text{ where} \tag{139}$$

$$U_j = \{(r\cos(\theta), r\sin(\theta)) : \; r < 4+\sin(4\pi^2/\theta),\; \theta \in (2\pi/j, 2\pi]\} \tag{140}$$

$$V_j = \{(r\cos(\theta), r\sin(\theta)) : \; r < 4,\; \theta \in (0, 2\pi/j]\}. \tag{141}$$

As in Example 2.13, (M_j, d_{M_j}) have no Gromov-Haudorff limit because they have increasingly many splines. Unlike Example 2.13, for any number N, there exists δ_N sufficiently small that $M_j^{\delta_N}$ has N splines. In fact,

$$(M_j^\delta, d_{M_j}) \xrightarrow{GH} (Y^\delta, d_Y) \tag{142}$$

where Y^δ is δ inner region of the flat open manifold:

$$Y = \{(r\cos(\theta), r\sin(\theta)) : \; r < 4+\sin(4\pi^2/\theta),\; \theta \in (0, 2\pi]\} \tag{143}$$

Taking the identity maps to be the isometric embeddings, we see that Y is also a glued limit space for the M_j even though it is bounded but not precompact.

Recall Example 2.14 of a sequence of surfaces which have no Gromov-Hausdorff limit. We now modify this example to obtain a sequence of manifolds with boundary that have no Gromov-Haudorff limit but whose δ inner regions have Gromov-Haudorff limits and we construct the glued limit space and see that it is also bounded and not precompact. This glued limit space is not a manifold.

Example 6.12. *Let*

$$X_j = ([0,1]\times[0,1]) \sqcup ([0,1]\times[0,1/2]) \sqcup \cdots \sqcup ([0,1]\times[0,1/2^j]) \tag{144}$$

be a disjoint union of spaces with taxicab metrics glued with a gluing map $\psi(0,y) = (0,y)$. One may think of X_j as a book with j pages of decreasing height glued along a spine on the left. Within X_j choose sets A_j to be the union of the top edges of each of the pages. If we take surfaces M_j as constructed in Proposition 2.8 they now have boundary, such that

$$d_{GH}(M_j, X_j) \to 0 \text{ and } d_{GH}(M_j^\delta, X_j \setminus T_\delta(A_j)) \to 0. \tag{145}$$

As in Example 2.14, the M_j have no GH converging subsequence because the X_j have no GH converging subsequence.

Observe that there exists k_δ such that for all $j > k_\delta$,

$$X_j \setminus T_\delta(A_j) = ([0,1]\times[0,\delta)) \sqcup ([0,1]\times[0,1/2-\delta]) \sqcup \cdots \sqcup ([0,1]\times[0,1/2^{k_\delta}-\delta]). \tag{146}$$

Since this sequence does not depend on j, it clearly converges in the Gromov-Hausdorff sense. Thus M_j^δ converge to the same Gromov-Hausdorff limit space. In fact they converge to $X_\infty \setminus T_\delta(A_\infty)$ where

$$X_\infty = ([0,1]\times[0,1)) \sqcup ([0,1]\times[0,1/2]) \sqcup \cdots \sqcup ([0,1]\times[0,1/2^j])\cdots \tag{147}$$

and A_∞ is the union of the tops of all of these pages. In fact, X_∞ is the glued limit space.

6.5. **Glued Limit Space is not Geodesic.** Here we present an example whose glued limit space is not geodesic or even a length space:

Remark 6.13. *In Euclidean space, $\mathbb{E}^2$, define*

$$M_j = (-1,1)\times(-1,1)\setminus[-1/2,1/2]\times[0,1-1/j]. \tag{148}$$

Then for $\delta < 1/4$ there is $J = J(\delta)$ such that

$$M_j^\delta = ((-1+\delta,1-\delta)\times(-1+\delta,-\delta))\sqcup((-1+\delta,-1/2-\delta)\times(-\delta,1-\delta))\sqcup((1/2-\delta,1-\delta,)\times(-\delta,1-\delta)) \tag{149}$$

for $j \ge J$.

Thus $\bar{M}_j^\delta$ is a constant sequence for $j \ge J$ and $\bar{M}_j^\delta \xrightarrow{GH} Y^\delta$ where

$$Y^\delta = [-1+\delta,1-\delta]\times[-1+\delta,0]\cup[-1+\delta,-1/2+\delta]\times[0,1-\delta]\cup[1/2-\delta,1-\delta]\times[0,1-\delta]. \tag{150}$$

The glued-limit,

$$Y = \bigcup Y^\delta = [-1,1]\times[-1,0]\ \cup\ [-1,-1/2]\times[0,1]\ \cup\ [1/2,1]\times[0,1]\subset\mathbb{E}^2 \tag{151}$$

is not a length space. See that $\bar{M}_j \xrightarrow{GH} X = Y\cup(\{1\}\times[-1/2,1/2])$.

Open Question 6.14. *Is a glued limit space locally geodesic: for all $y \in Y$, does there exist $\epsilon_y > 0$ such that $B(y,\epsilon_y)$ is geodesic? If there is a counter example, what conditions can be imposed on the space to guarantee that it is locally geodesic?*

6.6. **Balls in Glued Limit Spaces.** Recall that earlier we proved that for any $p \in M^{\delta_i}$, if $x \in B_p(\delta_i - \delta_{i+1}) \in M$, then $x \in M^{\delta_{i+1}}$ [Lemma 3.3]. This is not true for glued limit spaces. That is, it is possible for $p \in F_{\delta_i}(Y^{\delta_i})$ to have an $x \in B_p(\delta_i - \delta_{i+1}) \in Y$ such that $x \notin F_{\delta_{i+1}}(Y^{\delta_{i+1}})$. In fact we can take the ball of arbitrarily small radius and still $x \notin F_{\delta_{i+1}}(Y^{\delta_{i+1}})$.

Here we present such an example:

Example 6.15. *Recall Example 6.12 where we constructed M_j that have no Gromov-Hausdorff limit such that M_j^δ converge in the Gromov-Hausdorff sense to $Y^\delta = X_\infty\setminus T_\delta(A_\infty)$ where*

$$X_\infty = ([0,1]\times[0,1])\sqcup([0,1]\times[0,1/2])\sqcup\cdots\sqcup([0,1]\times[0,1/2^j])\cdots \tag{152}$$

where each piece is connected along $(0,y)\sim(0,y)$ and A_∞ is the union of the tops of all of these pages. This X_∞ is a glued limit space for this example.

Then $F_\delta(Y^\delta) = X_\infty\setminus T_\delta(A_\infty)$. Take any ball about the common point $(0,0)\in X_\infty$. For any radius $r > 0$, $B_{(0,0)}(r)$ contains infinitely many points $y_j = (r/2,0)\in[0,1]\times[0,1/2^j]$. However $y_j \notin F_\delta(Y^\delta)$, for j sufficiently large that $1/2^j < \delta$.

6.7. **Nonuniqueness of the Glued Limit Space.** We now see that glued limit spaces are not necessarily unique. Recall that in Remark 6.7 we explained that if M_j have a Gromov-Hausdorff limit, then the glued limit space is unique. So we need to construct a sequence of manifolds, M_j, which have no Gromov-Hausdorff limit. In fact we will imitate Example 6.12 applying Proposition 2.8 to construct the following example:

Example 6.16. *There is a sequence of Riemannian surfaces, M_j, with boundary, ∂M_j, such that there exists $\delta_i \to 0$ and metric spaces Y^{δ_i} such that*

$$d_{GH}(M_j^{\delta_i}, Y^{\delta_i}) \to 0 \tag{153}$$

yet there are two different glued limit spaces $Y_1 = Y(\delta_{2i},\varphi_{\delta_{2i},\delta_{2i+2}})$ and $Y_2 = Y(\delta_{2i},\varphi'_{\delta_{2i},\delta_{2i+2}})$ constructed as in Theorem 6.3 and Theorem 6.1.

Proof. Let

$$P_1 = [0,1]\times[-1/2,1/2] \tag{154}$$
$$P_2 = [0,1]\times[-1/4,1/4] \tag{155}$$
$$P_3 = [0,1]\times[-1/6,1/6] \tag{156}$$
$$P_j = [0,1]\times[-1/(2j),1/(2j)] \tag{157}$$

and let

$$X_j = P_1 \sqcup P_2 \sqcup P_2 \tag{158}$$
$$\sqcup P_3 \sqcup P_3 \sqcup P_3 \sqcup P_3 \tag{159}$$
$$\sqcup P_4 \sqcup P_4 \sqcup P_4 \sqcup P_4 \sqcup P_4 \sqcup P_4 \sqcup P_4 \sqcup P_4 \tag{160}$$
$$\sqcup P_5 \sqcup \cdots \sqcup P_5 \tag{161}$$
$$\sqcup \cdots \sqcup P_j \sqcup \cdots \sqcup P_j \tag{162}$$

be a disjoint union of $N_j = 1 + 2 + 4 + ... + 2^{j-1}$ spaces endowed with taxicab metrics glued with a gluing map $\psi(0,y) = (0,y)$. One may think of X_j as a book with N_j pages of different heights glued along a spine on the left.

Let $H_j \subset P_j$ be defined by

$$H_j = [0,1]\times\{-1/2j\} \cup \{1\}\times[-1/(2j),1/(2j)] \cup [0,1]\times\{1/(2j)\} \subset P_j \tag{163}$$

and let $A_j \subset X_j$ be defined,

$$A_j = H_1 \sqcup H_2 \sqcup H_2 \tag{164}$$
$$\sqcup H_3 \sqcup H_3 \sqcup H_3 \sqcup H_3 \tag{165}$$
$$\sqcup H_4 \sqcup H_4 \sqcup H_4 \sqcup H_4 \sqcup H_4 \sqcup H_4 \sqcup H_4 \sqcup H_4 \tag{166}$$
$$\sqcup H_5 \sqcup \cdots \sqcup H_5 \tag{167}$$
$$\sqcup \cdots \sqcup H_j \sqcup \cdots \sqcup H_j. \tag{168}$$

If we take surfaces M_j as constructed in Proposition 2.8 they now have boundary, such that

$$d_{GH}(M_j, X_j) \to 0 \text{ and } d_{GH}(M_j^\delta, X_j \setminus T_\delta(A_j)) \to 0. \tag{169}$$

As in Example 2.14, the M_j have no GH converging subsequence because the X_j have no GH converging subsequence.

Now

$$X_j \setminus T_\delta(A_j) = (P_1 \setminus T_\delta(H_1)) \sqcup (P_2 \setminus T_\delta(H_2)) \sqcup (P_2 \setminus T_\delta(H_2)) \tag{170}$$
$$\sqcup (P_3 \setminus T_\delta(H_3)) \sqcup \cdots \sqcup (P_3 \setminus T_\delta(H_3)) \tag{171}$$
$$\sqcup \cdots \sqcup (P_j \setminus T_\delta(H_j)) \sqcup \cdots \sqcup (P_j \setminus T_\delta(H_j)). \tag{172}$$

Observe that

$$P_j \setminus T_\delta(H_j) = [0, 1-\delta]\times[-1/(2j)\ +\delta, 1/(2j)\ -\delta] \tag{173}$$

Taking $\delta = \delta_{2i} = 1/(2i)$ and $j > i$ we have

$$P_i \setminus T_\delta(H_i) = [0, 1-1/(2i)]\times\{0\} \tag{174}$$

and

$$P_j \setminus T_\delta(H_j) = \emptyset \tag{175}$$

Thus

$$X_j \setminus T_\delta(A_j) = (P_1 \setminus T_\delta(H_1)) \sqcup (P_2 \setminus T_\delta(H_2)) \sqcup (P_2 \setminus T_\delta(H_2)) \tag{176}$$
$$\sqcup (P_3 \setminus T_\delta(H_3)) \sqcup \cdots \sqcup (P_3 \setminus T_\delta(H_3)) \tag{177}$$
$$\sqcup \cdots \sqcup (P_{i-1} \setminus T_\delta(H_{i-1}) \sqcup \cdots \sqcup (P_{i-1} \setminus T_\delta(H_{i-1})) \tag{178}$$
$$\sqcup [0, 1 - 1/(2i)] \times \{0\} \sqcup \cdots \sqcup [0, 1 - 1/(2i)] \times \{0\} \tag{179}$$

endowed with taxicab metrics glued with a gluing map $\psi(0, y) = (0, y)$. There are $1 + 2 + 4 + ... + 2^{(i-1)-1}$ rectangular pages and $2^{(i-1)}$ pages that are just intervals of length $1-1/(2i)$. Taking $j \to \infty$ we get

$$d_{GH}(X_j \setminus T_\delta(A_j), Y^\delta) \to 0 \tag{180}$$

where

$$Y^{\delta_{2i}} = Y^{1/(2i)} = X_j \setminus T_{\delta_{2i}}(A_j) \qquad \forall j > i. \tag{181}$$

So

$$Y^{\delta_{2i}} = (P_1 \setminus T_{1/(2i)}(H_1)) \sqcup (P_2 \setminus T_{1/(2i)}(H_2)) \sqcup (P_2 \setminus T_{1/(2i)}(H_2)) \tag{182}$$
$$\sqcup (P_3 \setminus T_{1/(2i)}(H_3)) \sqcup \cdots \sqcup (P_3 \setminus T_{1/(2i)}(H_3)) \tag{183}$$
$$\sqcup \cdots \sqcup (P_{2i-1} \setminus T_{1/(2i)}(H_{2i-1}) \sqcup \cdots \sqcup (P_{2i-1} \setminus T_{1/(2i)}(H_{2i-1})) \tag{184}$$
$$\sqcup [0, 1 - 1/(2i)] \times \{0\} \sqcup \cdots \sqcup [0, 1 - 1/(4i)] \times \{0\} \tag{185}$$

endowed with taxicab metrics glued with a gluing map $\psi(0, y) = (0, y)$ where there are $1 + 2 + 4 + ... + 2^{(i-1)-1}$ rectangular pages and $2^{(i-1)}$ pages that are just intervals of length $1 - 1/(2i)$.

If we define $\varphi_{\delta_{2i},\delta_{2i+2}} : Y^{\delta_{2i}} \to Y^{\delta_{2i+2}}$ to be the inclusion map, and then construct the glued limit space as in Theorem 6.1 we obtain,

$$Y_1 = Y(\delta_{2i}, \varphi_{\delta_{2i},\delta_{2i+2}}) = Y = (P_1 \setminus H_1) \sqcup (P_2 \setminus H_2) \sqcup (P_2 \setminus H_2) \tag{186}$$
$$\sqcup (P_3 \setminus H_3) \sqcup \cdots \sqcup (P_3 \setminus H_3) \cdots \tag{187}$$
$$\cdots \sqcup (P_j \setminus H_j) \sqcup \cdots \sqcup (P_j \setminus H_j) \cdots \tag{188}$$

endowed with taxicab metrics glued with a gluing map $\psi(0, y) = (0, y)$. This has infinitely many pages, all shaped like rectangles.

Now we define $\varphi'_{\delta_{2i},\delta_{2i+2}} : Y^{\delta_{2i}} \to Y^{\delta_{2i+2}}$ to be an isometric embedding which maps a point

$$(x, y) \in P_k \setminus T_{\delta_{2i}}(H_k) \subset Y^{\delta_{2i}} \tag{189}$$

for $k < i$ to

$$(x, y) \in P_k \setminus T_{\delta_{2i+2}}(H_k) \subset Y^{\delta_{2i+2}} \tag{190}$$

via the inclusion map and which maps

$$(x, y) \in P_i \setminus T_{\delta_{2i}}(H_i) = [0, 1 - 1/(2i)] \times \{0\} \subset Y^{\delta_{2i}} \tag{191}$$

to

$$(x, y - \delta_{2i} + \delta_{2i+2}) \in P_{i+1} \setminus T_{\delta_{2i+2}}(H_{i+1}) = [0, 1 - 1/(2i + 2)] \times \{0\} \subset Y^{\delta_{2i+2}}. \tag{192}$$

This is possible because we have enough copies of $P_{i+1} \setminus T_{\delta_{2i+2}}(H_{i+1})$ in $Y^{\delta_{2i+2}}$.

In particular $\varphi'_{\delta_{2i},\delta_{2i+2}}$ maps the interval pages into interval pages. If we then construct the glued limit space as in Theorem 6.1 we obtain,

$$Y_2 = Y(\delta_{2i}, \varphi'_{\delta_{2i},\delta_{2i+2}}) = Y \sqcup [0, 1] \times \{0\} \sqcup [0, 1] \times \{0\} \sqcup [0, 1] \times \{0\} \sqcup \cdots \tag{193}$$

which has infinitely many pages that are intervals in addition to all the pages shaped like rectangles. So we have two distinct glued limit spaces for the sequence $\delta_{2i} = 1/(2i)$. □

7. Glued Limits under Curvature Bounds

In this section we prove the existence of glued limits of sequences of manifolds with certain natural geometric conditions [Theorems 7.1 and 7.4]. We do not require the sequences of manifolds themselves to have Gromov-Hausdorff limits.

7.1. **Constructing Glued Limits of Manifolds with Constant Sectional Curvature.**

In this section we prove that if $M_j \in \mathcal{M}_H^{m,V,l}$ (see Definition 5.1) then the sequence has a glued limit space [Theorem 7.1]. The sequence need not have a Gromov-Hausdorff limit (see Remark 7.2).

Theorem 7.1. *Given any $\delta_0 > 0$ if $(M_j, g_j) \subset \mathcal{M}_H^{m,V,l}$, then there is a Gromov-Hausdorff convergent subsequence $\{M_{j_k}^{\delta_0}\}$ and a glued-limit space Y such that for all $\delta \in (0, \delta_0]$ there exists a further subsequence $\{j'_k\}$ of $\{j_k\}$ for which $M_{j'_k}^{\delta}$ converges in Gromov-Hausdorf sense to a compact metric space Y^δ and for any such Y^δ there exists an isometric embedding*

$$F_\delta : Y^\delta \to Y. \tag{194}$$

Remark 7.2. *The sequences of flat surfaces, $M_j \subset \mathbb{E}^2$, defined in Example 2.13 and Example 6.11 have a common finite upper volume bound but there is no common finite upper bound for the number of disjoint balls of M_j of radius less than 1. Thus, these two sequences do not have a Gromov-Hausdorff limit. Nonetheless since*

$$L_{min}(M_j) = \inf\{L_g(C) : \text{ C is a closed geodesic in M_j }\} > l \tag{195}$$

Theorem 7.1 demonstrates that we can construct glued limits for these spaces.

Remark 7.3. *The choice of the a further subsequence $\{j'_k\}$ of $\{j_k\}$ in Theorem 7.1 is necessary. Let $(M_j, g_j) \subset \mathcal{M}_0^{2,V,l}$ be the sequence defined in defined in Example 4.4. Take $\delta_0 = 3\varepsilon$. Then $\{M_j^{\delta_0}\}$ is a Gromov-Hausdorffconvergent sequence. Choosing $2\varepsilon \in (0, \delta_0]$ we see that $\bar{M}_{2j}^{2\varepsilon}$ converges in Gromov-Hausdorff sense but $\bar{M}_j^{2\varepsilon}$ does not.*

Proof. Consider the sequence δ_0, $\delta_i = \delta_0/i$, $i \in \mathbb{N}$. Start with δ_0. By Theorem 5.2 there exist a subsequence $\{j_k(\delta_0)\}_{k=1}^\infty$ of $\{j\}_{j=1}^\infty$ and a compact metric space Y^{δ_0} such that

$$\left(\bar{M}_{j_k(\delta_0)}, d_{M_{j_k(\delta_0)}}\right) \xrightarrow{\text{GH}} \left(Y^{\delta_0}, d_{Y^{\delta_0}}\right). \tag{196}$$

Proceeding as before for $n \in \mathbb{N}$, there is a subsequence $\{j_k(\delta_n)\}_{k=1}^\infty$ of $\{j_k(\delta_{n-1})\}_{k=1}^\infty$ and a compact metric space $Y^{\delta_n}(j_k(\delta_n))$ such that

$$(\bar{M}_{j_k(\delta_n)}^{\delta_n}, d_{M_{j_k(\delta_n)}}) \xrightarrow{\text{GH}} Y^{\delta_n}. \tag{197}$$

Define $j_k = j_k(\delta_k)$. We have

$$(\bar{M}_{j_k}^{\delta_n}, d_{M_{j_k}}) \xrightarrow{\text{GH}} Y^{\delta_n} \tag{198}$$

for $n = 0, 1, 2, \cdots$ since $\{j_k\}_{k=n}^\infty$ is a subsequence of $\{j_k(\delta_n)\}_{k=1}^\infty$. We may now apply Theorem 6.3 to complete the proof. □

7.2. **Constructing Glued Limits with Ricci curvature bounds.** Here we prove that glued limits exist for noncollapsing sequences of manifolds with nonnegative Ricci curvature and bounded volume which have control on the intrinsic diameters of their inner regions (defined in (6):

Theorem 7.4. *Given $m \in \mathbb{N}$, a decreasing sequence, $\delta_i \to 0$, $i = 0, 1, 2, \cdots$, $V > 0$, $\theta > 0$, and $D_i > 0$, let (M_j, g_j) be a sequence of m dimensional open Riemannian manifolds with nonnegative Ricci curvature, $\mathrm{Vol}(M) \le V$, such that*

$$sup\Big\{ \mathrm{Diam}\Big(M_j^{\delta_i},\, d_{M_j^{\delta_i}}\Big) : \;\; j \in \mathbb{N}\Big\} < D_i \qquad \forall i \in \mathbb{N}, \tag{199}$$

and

$$\forall j \in \mathbb{N}\ \exists q_j \in M_j^{\delta_0} \text{ such that } \mathrm{Vol}(B_{q_j}(\delta_0)) \ge \theta\delta_0^m. \tag{200}$$

Then there exists a subsequence j_k such that for all δ_i $\{M_{j_k}^{\delta_i}\}$ converge in the Gromov-Hausdorff sense to a compact metric space Y^{δ_i}. Thus M_{j_k} have a glued-limit space Y such that for all $\delta \in (0, \delta_0]$ there exists a further subsequence $\{j'_k\}$ of $\{j_k\}$ for which $M_{j'_k}^{\delta}$ converges in Gromov-Hausdorff sense to a compact metric space Y^δ and for any such Y^δ there exists an isometric embedding

$$F_\delta : Y^\delta \to Y. \tag{201}$$

Remark 7.5. *If there is $D > 0$ such that*

$$sup_{\delta\in(0,\delta_0]}\Big\{ \mathrm{Diam}\,(M_j^\delta, d_{M_j^\delta})\Big\} \le D \tag{202}$$

Then we could take $D_i = D$ for all i. But this requirement is unnecessarily strong.

Remark 7.6. *The choice of a further subsequence $\{j'_k\}$ of $\{j_k\}$ in Theorem 7.1 is necessary. For the sequence (M_j, g_j) defined in Example 4.4, consider a decreasing sequence, $\delta_i \to 0$, $i = 0, 1, 2, \cdots$ such that $\delta_0 = 3\varepsilon$ and $\delta_1 = \varepsilon$. Then the hypotheses of the theorem are satisfied. For all δ_i, $\{M_j^{\delta_i}\}$ converges in Gromov-Hausdorff sense. Now for $2\varepsilon \in (0, \delta_0]$, $\{M_j^{2\varepsilon}\}$ does not have a Gromov-Hausdorff limit.*

Proof. Take $\delta \in (0, \delta_0]$, by hypothesis and Bishop-Gromov volume comparison theorem 2.19

$$\mathrm{Vol}(B_{q_j}(\delta)) \ge \mathrm{Vol}(B_{q_j}(\delta_0)) \left(\frac{\delta}{\delta_0}\right)^m \ge \theta\delta^m. \tag{203}$$

The above inequality and the hypotheses of the theorem imply that for each i,

$$\{(M_j, g_j)\} \subset \mathcal{M}_\theta^{m,\delta_i,D_i,V}. \tag{204}$$

Start with δ_0. By Theorem 1.4 there exists a subsequence $\{j_k(\delta_0)\}_{k=1}^\infty$ of $\{j\}_{j=1}^\infty$ such that

$$\Big(\bar{M}_{j_k(\delta_0)}^{\delta_0}, d_{M_{j_k(\delta_0)}}\Big) \xrightarrow{\text{GH}} \Big(Y^{\delta_0}, d_{Y^{\delta_0}}\Big). \tag{205}$$

Proceeding as before for $n \in \mathbb{N}$, there exists a subsequence $\{j_k(\delta_n)\}_{k=1}^\infty$ of $\{j_k(\delta_{n-1})\}_{k=1}^\infty$ and a compact metric space $Y^{\delta_n}(j_k(\delta_n))$ such that

$$\Big(\bar{M}_{j_k(\delta_n)}^{\delta_n}, d_{M_{j_k(\delta_n)}}\Big) \xrightarrow{\text{GH}} \Big(Y^{\delta_n}, d_{Y^{\delta_n}}\Big). \tag{206}$$

Define $j_k = j_k(\delta_k)$. We have

$$\Big(\bar{M}_{j_k}^{\delta_n}, d_{M_{j_k}}\Big) \xrightarrow{\text{GH}} \Big(Y^{\delta_n}, d_{Y^{\delta_n}}\Big) \tag{207}$$

for $n \in \mathbb{N}$ since $\{j_k\}_{k=n}^\infty$ is a subsequence of $\{j_k(\delta_n)\}_{k=1}^\infty$. Finally, apply Theorem 6.3. □

8. Properties of Glued Limit Spaces under Curvature Bounds

In this final section of the paper we consider the local properties of the glued limits of sequences of manifolds with constant sectional curvature as in Theorem 7.1 and manifolds with nonnegative Ricci curvature as in Theorem 7.4. We begin with an example indicating how even when the sequences of manifolds has a Gromov-Hausdorff limit, one need not retain curvature conditions on the Gromov-Haudorff limit space [Example 8.1]. This is in sharp contrast with the setting where the Riemannian manifolds are compact without boundary. In this example, the glued limit space is empty. Then we have a subsection about balls in glued limit spaces without any assumption on curvature [Theorem 8.3]. We apply this control on the balls to prove that local curvature properties do persist on glued limit space. In particular we prove Proposition 8.4 that the glued limits of manifolds with constant sectional curvature bounds (and other conditions) are unions of manifolds with constant sectional curvature. We close with Theorem 8.8 concerning the metric measure properties of glued limits of manifolds with nonnegative Ricci curvature.

8.1. **An Example with no Curvature Control.**

We now construct a sequence of flat open manifolds whose Gromov-Hausdorff limit is not flat:

Example 8.1. *Let $B_p(1) \subset \mathbb{H}^2$ be a unit ball in hyperbolic space and $B_0(1) \subset \mathbb{E}^2$ be the unit ball in Euclidean space. Then $\exp_p : B_0(1) \to B_p(1)$. Let*

$$S_j = \{(i/j, k/j) : i, k \in \mathbb{Z}\} \cap B_0(1) \subset \mathbb{E}^2 \tag{208}$$

and $S'_j = \exp_p(S_j)$. We can form a graph A_j whose vertices are in S_j and whose edges form a triangulation. That is we connect $(i/j, k/j)$ to the points $((i+1)/j, k/j)$, $(i/j, (k+1)/j)$ and $((i+1)/j, (k+1)/j)$. We let $A'_j = \exp_p(A_j)$ and set the lengths of the edges in A'_j to be the distances between the vertices viewed as points in $\mathbb{H}^2$. Then A'_j converges to $B_p(1) \subset \mathbb{H}^2$ in the Gromov-Hausdorff sense.

Now we define A''_j to be the simplicial complexes formed by filling in the triangles in A'_j with flat Euclidean triangles. Observe that A''_j converges to $B_p(1) \subset \mathbb{H}^2$ in the Gromov-Hausdorff sense as well. Finally, for each j we remove tiny balls of radius $<< 1/j$ around the vertices in A''_j, to create a flat open manifold, M_j. These M_j converge in the Gromov-Hausdorff sense to $B_p(1) \subset \mathbb{H}^2$.

Remark 8.2. *Example 8.1 has an empty glued limit space. In the next subsections we will see that the glued limit spaces do retain some of the curvature properties of the initial sequence of manifolds. Thus the glued limit space is a more natural object of study than the Gromov-Haudorff limit even when the Gromov-Haudorff limit exists.*

8.2. **Balls to Glued Limit Spaces.**

Generally when one wishes to study the properties of a complete noncompact limit space, one studies balls in the limit space as Gromov-Hausdorff limits of balls in the sequence. Here we cannot control balls in the limit space, but we can control balls of radius $\epsilon < \delta_i - \delta_{i+1}$ centered in $F_{\delta_i}(Y^{\delta_i})$ intersected with $F_{\delta_{i+1}}(Y^{\delta_{i+1}})$. This will suffice to study the geometric properties of the glued limit spaces.

Theorem 8.3. *Let Y be a glued limit of a sequence of Riemannian manifolds M_j as in Theorem 6.3. If $y \in Y^{\delta_i}$ and $\epsilon < \delta_i - \delta_{i+1}$, then there exists a subsequence $M^{\delta_i}_{j_k}$ containing points y_{j_k} and $\epsilon_{j_k} \to \epsilon$ such that*

$$B(y_{j_k}, \epsilon_{j_k}) = \left\{x \in M_{j_k} : d_M(x, y_{j_k}) < \epsilon_{j_k}\right\} \subset M^{\delta_{i+1}}_{j_k}. \tag{209}$$

and

$$d_{GH}\Big(\Big(\bar{B}(y_{j_k},\epsilon_{j_k}),d_{M_{j_k}}\Big),\Big(\bar{B}(F_{\delta_i}(y),\epsilon)\cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}),d_Y\Big)\Big)\xrightarrow{GH}0. \tag{210}$$

Note that in Example 6.15 we saw that $B(F_{\delta_i}(y),\epsilon)\cap F_{\delta_{i+1}}(Y^{\delta_{i+1}})$ need not be isometric to $B(F_{\delta_i}(y),\epsilon)\subset Y$ even when ϵ is taken arbitrarily small.

Proof. Recall that $\varphi_{\delta_{i+1},\delta_i}$ was defined in the following way, see Theorem6.1. We picked isometric embeddings

$$\varphi_j : M_j^{\delta_{i+1}}\to Z, \tag{211}$$

$$\varphi_\infty : Y^{\delta_{i+1}}\to Z \tag{212}$$

such that

$$d_H^Z\Big(\varphi_j(M_j^{\delta_{i+1}}),\varphi_\infty(Y_j^{\delta_{i+1}})\Big)\to 0 \tag{213}$$

Then we found a subsequence such that

$$d_H^Z\Big(\varphi_{j_k}(M_{j_k}^{\delta_i}),X^{\delta_i}\Big)\to 0 \tag{214}$$

and chose $\varphi_{\delta_{i+1},\delta_i}$ to be an isometry such that

$$\varphi_{\delta_{i+1},\delta_i}(Y^{\delta_i})=\varphi_\infty^{-1}(X^{\delta_i}). \tag{215}$$

Then there exist

$$y_{j_k}\in M_{j_k}^{\delta_i}\subset M_{j_k}^{\delta_{i+1}}\subset M_{j_k} \tag{216}$$

such that

$$d_H^Z\Big(\varphi_{j_k}(y_{j_k}),\varphi_\infty(\varphi_{\delta_{i+1},\delta_i}(y)\Big)\to 0 \tag{217}$$

Let $\epsilon'\in(0,\delta_i-\delta_{i+1})$. Then by Lemma 3.3 we have

$$B(y_{j_k},\epsilon')=\Big\{x\in M_{j_k}:d_M(x,y_{j_k})<\epsilon'\Big\}\subset M_{j_k}^{\delta_{i+1}}. \tag{218}$$

From this and since $\varphi_{j_k}:M_{j_k}^{\delta_{i+1}}\to Z$ is an isometry into its image:

$$\Big(B(y_{j_k},\epsilon'),d_{M_{j_k}^{\delta_{i+1}}}\Big)\ \text{ is isometric to }\ \Big(B(\varphi_{j_k}(y_{j_k}),\epsilon')\cap\varphi_{j_k}(M_{j_k}^{\delta_{i+1}}),d_Z\Big). \tag{219}$$

By Lemma 2.2, for any $\epsilon\in(0,\delta_i-\delta_{i+1})$, there exists $\epsilon_{j_k}\to\epsilon$ eventually in $(0,\delta_i-\delta_{i+1})$, such that

$$\bar{B}\Big(\varphi_{j_k}(y_{j_k}),\epsilon_{j_k}\Big)\cap\varphi_{j_k}\Big(M_{j_k}^{\delta_{i+1}}\Big)\ \xrightarrow{\text{H}}\ \bar{B}\Big(\varphi_\infty\varphi_{\delta_{i+1,\delta_i}}(y),\epsilon\Big)\cap\varphi_\infty\Big(Y^{\delta_{i+1}}\Big). \tag{220}$$

Now,

$$\Big(\bar{B}(\varphi_\infty\varphi_{\delta_{i+1},\delta_i}(y),\epsilon)\cap\varphi_\infty(Y^{\delta_{i+1}}),d_Z\Big) \tag{221}$$

is isometric to

$$\Big(\bar{B}(\varphi_{\delta_{i+1},\delta_i}(y),\epsilon),d_{Y^{\delta_{i+1}}}\Big) \tag{222}$$

which is isometric to

$$\Big(F_{\delta_{i+1}}\bar{B}(\varphi_{\delta_{i+1},\delta_i}(y),\epsilon),d_{F_{\delta_{i+1}}Y^{\delta_{i+1}}}\Big) \tag{223}$$

which is isometric to

$$\Big(\bar{B}(F_{\delta_{i+1}}\varphi_{\delta_{i+1},\delta_i}(y),\epsilon)\cap F_{\delta_{i+1}}Y^{\delta_{i+1}},d_Y\Big). \tag{224}$$

Hence

$$d_{GH}\Big(\big(\bar{B}(y_{j_k},\epsilon_{j_k}), d_{M_{j_k}}\big), \big(\bar{B}(F_{\delta_{i+1}}(\varphi_{\delta_{i+1},\delta_i}(y)),\epsilon)\cap F_{\delta_{i+1}}Y^{\delta_{i+1}}, d_Y\big)\Big) \xrightarrow{\text{GH}} 0. \tag{225}$$

□

8.3. **Properties of Glued Limits of Manifolds with Constant Sectional Curvature.** Here we prove a proposition, present a key example and state two open questions concerning the glued limits of manifolds with constant sectional curvature.

Proposition 8.4. *Let Y be a glued limit space obtained from a sequence $M_j \in \mathcal{M}_H^{m,V,l}$ as in Theorem 7.1. Then there exists a countable collection of sets, $W_i \subset Y$, each of which is isometric to an m dimensional smooth open manifold of constant sectional curvature, H, such that*

$$Y \subset \bigcup_{i=1}^{\infty} W_i. \tag{226}$$

In fact

$$F_{\delta_i}(Y^{\delta_i}) \subset W_i \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y. \tag{227}$$

See Example 8.5 in which the glued limit space is a countable collection of flat tori which are not connected to one another but have a metric restricted from a larger compact metric space of finite volume.

Proof. Recall that any glued limit space, Y, defined as in Theorem 6.3 depends on a sequence $\delta_i \to 0$ and gluings $\varphi_{\delta_{i+1},\delta_i} : Y^{\delta_{i+1}} \to Y^{\delta_i}$. There are isometric embeddings $F_{\delta_i} : Y^{\delta_i} \to Y$ such that

$$Y \subset \bigcup_{i=1}^{\infty} F_{\delta_i}(Y^{\delta_i}) \tag{228}$$

and

$$F_{\delta_i}(Y^{\delta_i}) \subset F_{\delta_{i+1}}(Y^{\delta_{i+1}}). \tag{229}$$

Let

$$\epsilon_i = \min\left\{\delta_i - \delta_{i+1}, l/2, \pi\sqrt{h}/2\right\}/2 \tag{230}$$

where $h = H$ when $H > 0$ and $h = (l/\pi)^2$ otherwise.

Let

$$W_i = T_{\epsilon_i}\Big(F_{\delta_i}(Y^{\delta_i})\Big) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y \tag{231}$$

First observe that by (229) we have

$$F_{\delta_i}(Y^{\delta_i}) \subset W_i. \tag{232}$$

So combined with (228), we have (226). So we need only show W_i is a smooth m dimensional open manifold of constant sectional curvature, H.

For all $w \in W_i$, there exists $y_\infty \in F_{\delta_i}(Y^{\delta_i})$ such that $w \in B_{y_\infty}(\epsilon_i) \subset Y$. Since

$$B_{y_\infty}(\epsilon_i) \subset T_{\epsilon_i}\Big(F_{\delta_i}(Y^{\delta_i})\Big) \tag{233}$$

we have

$$U = B_{y_\infty}(\epsilon_i) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) = B_{y_\infty}(\epsilon_i) \cap W_i. \tag{234}$$

We need only show that U is isometric to a ball of radius ϵ_i in M_H^m, the m dimensional simply connected manifold with constant sectional curvature H.

There exists $y \in Y^{\delta_i}$ such that $y_\infty = F_{\delta_i}(y)$. By Theorem 8.3, and the fact that $\epsilon_i < \delta_i - \delta_{i+1}$, there exists a subsequence $M^{\delta_i}_{j_k}$ containing points y_{j_k} and $\epsilon_{j_k} \to \epsilon_i$ such that (209) and (210) are satisfied.

Since $\epsilon_i < l/2$, then for k sufficiently large $\epsilon_{j_k} < l/2$ and so by (209) and M_j satisfy the conditions of Theorem 5.2, by (90) we know there is an Riemannian isometric diffeomorphism from $B(y_{j_k}, \epsilon_{j_k})$ to a ball in M^m_H, the m dimensional simply connected manifold with constant sectional curvature H. Since $\epsilon_i < \sqrt{H}\pi/2$ when $H > 0$, we have a convex ball, so that, as metric spaces,

$$(B(y_{j_k}, \epsilon_{j_k}), d_M) \text{ is isometric to } (B(p, \epsilon_{j_k}), d_{M^m_H}). \tag{235}$$

Taking $k \to \infty$, the closure of these latter balls converge in the Gromov-Hausdorff sense to $(\bar{B}(p, \epsilon_i), d_{M^m_H})$. Thus by (210) and the uniqueness of Gromov-Hausdorff limits,

$$(\bar{B}(y_\infty, \epsilon_i) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}), d_Y) \text{ is isometric to } (\bar{B}(p, \epsilon_i), d_{M^m_H}). \tag{236}$$

Thus we have (234) and we are done. □

Example 8.5. *In this example we construct a glued limit space, Y, for a sequence of manifolds, M^m_j, satisfying the conditions of Theorem 5.2. In addition the M^m_j converge in the GH sense to a metric space X, so that the glued limit space is unique. The glued limit Y is a countable union of connected flat manifolds with the restricted metric from X.*

Proof. Let M_1 be two flat square annuli connected by a slanted strip of width 1 and length $\sqrt{2}$:

$$M_1 = C_{0,1} \cup C_{1,1} \cup S_{0,1} \subset \mathbb{R}^3 \tag{237}$$

where

$$C_{0,1} = \Big(((-1,1)\times(-1,1)) \setminus ((-1/2,1/2)\times[-1/2,1/2])\Big)\times\{0\} \tag{238}$$

$$C_{1,1} = \Big(((-1,1)\times(-1,1)) \setminus ((-1/2,1/2)\times[-1/2,1/2])\Big)\times\{1\} \tag{239}$$

$$S_{0,1} = \Big\{(x,y,z) : \ (x,y)\in(-1/2,1/2)\times[-1/2,1/2], \ z = x+1/2\Big\}. \tag{240}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that for $\delta > 1/4$,

$$M_1^\delta \subset C_{0,1} \cup C_{1,1}. \tag{241}$$

Let M_2 be three flat square annuli of total area $\le 4+4+4(1/4)$ connected by two slanted strips of width $1/2$:

$$M_2 = C_{0,2} \cup C_{1,2} \cup C_{2,2} \cup S_{1,2} \cup S_{2,2} \subset \mathbb{R}^3 \tag{242}$$

where

$$C_{0,2} = \Big(((-1,1)\times(-1,1)) \setminus ((-1/4,1/4)\times[-1/4,1/4])\Big)\times\{0\} \tag{243}$$

$$C_{1,2} = \Big(((-1/2,1/2)\times(-1/2,1/2)) \setminus ((-1/4,1/4)\times[-1/4,1/4])\Big)\times\{1/2\} \tag{244}$$

$$C_{2,2} = \Big(((-1,1)\times(-1,1)) \setminus ((-1/4,1/4)\times[-1/4,1/4])\Big)\times\{2/2\} \tag{245}$$

$$S_{0,2} = \Big\{(x,y,z) : \ (x,y)\in(-1/4,1/4)\times[-1/4,1/4] \ z = x+1/4\Big\} \tag{246}$$

$$S_{1,2} = \Big\{(x,y,z) : \ (x,y)\in(-1/4,1/4)\times[-1/4,1/4] \ z = x+3/4\Big\}. \tag{247}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that for $\delta > 1/8$,

$$M_1^\delta \subset C_{0,2} \cup C_{1,2} \cup C_{2,2} \setminus (B((0,0),\delta)\times[0,1]). \tag{248}$$

Let M_j be $(j+1)$ flat square annuli of total area $\le 4+4\sum_{i=0}^{j}(1/2)^j$ connected by j slanted strips of width $(1/2^j)$:

$$M_j = \bigcup_{i=0}^{j} C_{i,j} \cup \bigcup_{i=0}^{j-1} S_{i,j} \subset \mathbb{R}^3 \tag{249}$$

where

$$\begin{aligned}
C_{0,j} &= \Big(((-1,1)\times(-1,1)) \setminus ((-1/2^{j+1},1/2^{j+1})\times[-1/2^{j+1},1/2^{j+1}])\Big)\times\{0\}\\
C_{i,j} &= \Big(((-2^{i-j},2^{i-j})\times(-2^{i-j},2^{i-j})) \setminus ((-1/2^{j+1},1/2^{j+1})\times[-1/2^{j+1},1/2^{j+1}])\Big)\times\{2^{i-j}\}\\
C_{j,j} &= \Big(((-1,1)\times(-1,1)) \setminus ((-1/2^{j+1},1/2^{j+1})\times[-1/2^{j+1},1/2^{j+1}])\Big)\times\{2^{j-j}\}\\
S_{0,j} &= \Big\{(x,y,z):\ (x,y)\in(-1/2^{j+1},1/2^{j+1})\times[-1/2^{j+1},1/2^{j+1}]\ z = x+1/2^{j+1}\Big\}\\
S_{i,j} &= \Big\{(x,y,z):\ (x,y)\in(-1/2^{j+1},1/2^{j+1})\times[-1/2^{j+1},1/2^{j+1}]\ z = m_j(x+1/2^{j+1})+2^{i-j}\Big\}
\end{aligned}$$

where

$$m_j = \frac{2^{i+1-j}-2^{i-j}}{(1/2^{j+1})-(-1/2^{j+1})}. \tag{250}$$

Endowed with the length metric, this is isometric to an open manifold with constant sectional curvature 0. Note that for $\delta > 1/2^{j+2}$,

$$M_j^\delta \subset C_{0,j} \cup \cdots \cup C_{j,j} \setminus (B((0,0),\delta)\times[0,1]). \tag{251}$$

The Gromov-Hausdorff limit of the M_j exists and can be see to be

$$X = \bigcup_{j=0}^{\infty} C_j \cup S_0 \subset \mathbb{R}^3 \tag{252}$$

where

$$C_0 = \Big(((-1,1)\times(-1,1))\Big)\times\{0\} \tag{253}$$

$$C_i = \Big(((-2^{i-j},2^{i-j})\times(-2^{i-j},2^{i-j}))\Big)\times\{2^{i-j}\} \tag{254}$$

$$S_0 = \Big\{(0,0,z):\ z\in[0,1]\Big\} \tag{255}$$

endowed with the length metric. The Gromov-Hausdorff limit, Y^δ of the M_j^δ exists and

$$Y^\delta \subset X \setminus (B((0,0),\delta)\times[0,1]). \tag{256}$$

In fact $Y = X \setminus S_0$. □

Open Question 8.6. *Are the glued limits of sequences of manifolds with constant sectional curvature open manifolds with constant sectional curvature? We know they need not be connected by Example 8.5.*

Open Question 8.7. *Are the glued limits of sequences of manifolds with constant sectional curvature unique? Perhaps an adaption of Example 8.5 could be applied to show that they are not.*

8.4. **Properties of Glued Limits of Manifolds with Nonnegative Ricci Curvature.** We now prove the final theorem of our paper and state the last two open questions:

Theorem 8.8. *Suppose we have a sequence of m dimensional open Riemannian manifolds M_j with nonnegative Ricci curvature and* $\mathrm{Vol}(M_j) \le V_0$ *and that there exists a sequence* $\delta_i \to 0$, *such that the inner regions,* $M_j^{\delta_i}$, *converge in the Gromov-Hausdorff sense as* $j \to \infty$ *to* Y^{δ_i} *without collapsing. Suppose that Y is a glued limit constructed as in Theorem 6.3. Then Y has Hausdorff dimension m,* $\mathcal{H}^m(Y) \le V_0$ *and its Hausdorff measure has positive lower density everywhere.*

Note that this theorem may be applied to study the glued limits of sequences of manifolds satisfying the conditions of Theorem 7.4.

To prove this theorem we will apply Cheeger-Colding's Volume Convergence Theorem [2][3] which was reviewed in Subsection 2.6. See Theorem 2.25 and Remark 2.26 for the precise statement we will use here.

Proof. First we prove that

$$W_i \;=\; T_{(\delta_i-\delta_{i+1})/2}\left(F_{\delta_i}\left(Y^{\delta_i}\right)\right) \;\cap\; F_{\delta_{i+1}}\left(Y^{\delta_{i+1}}\right) \;\subset\; Y. \tag{257}$$

have Hausdorff dimension m and have doubling Hasudorff measures. For any $w \in W_i$, let

$$U_w = B(w, (\delta_i - \delta_{i+1})/2) \cap W_i. \tag{258}$$

We can find $y \in Y^{\delta_i}$ such that $d_Y(y, w) < (\delta_i - \delta_{i+1})/2$. Then we have

$$U_w = B\left(w, (\delta_i - \delta_{i+1})/2\right) \cap B\left(F_{\delta_i}(y), \delta_i - \delta_{i+1}\right) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \tag{259}$$

By Theorem 8.3, we have a subsequence j_k, points $y_{j_k} \in M_{j_k}^{\delta_i}$ and $\epsilon_{j_k} \to \epsilon = (\delta_i - \delta_{i+1})/2$ satisfying (209) and (210):

$$d_{GH}\left(\left(\bar{B}(y_{j_k}, \epsilon_{j_k}), d_{M_{j_k}}\right), \left(\bar{B}(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}), d_Y\right)\right) \xrightarrow{\text{GH}} 0. \tag{260}$$

Combining this with the fact that

$$w \in B(y, (\delta_i - \delta_{i+1})/2) \subset \bar{B}(F_{\delta_i}(y), \epsilon) \cap F_{\delta_{i+1}}(Y^{\delta_{i+1}}) \subset Y \tag{261}$$

there exists

$$z_{j,k} \in \bar{B}(y_{j_k}, (\delta_i - \delta_{i+1})/2) \subset \bar{B}(y_{j_k}, \epsilon_{j_k}) \subset M_{j_k} \tag{262}$$

such that

$$d_{GH}\Big(\left(\bar{B}(z_{j_k}, (\delta_i - \delta_{i+1})/2), d_{M_{j_k}}\right), \left(\bar{U}_w, d_Y\right)\Big) \xrightarrow{\text{GH}} 0. \tag{263}$$

Since we assumed this is noncollapsing, then by the Cheeger-Colding Volume Convergence Theorem mentioned above we have

$$\mathcal{H}_m(B_w(r) \cap U_w) = \lim_{k\to\infty} \mathcal{H}_m(B_{z_{j_k}}(r)) \tag{264}$$

for all $r \le r_i = (\delta_i - \delta_{i+1})/2$. By (259) and Bishop's Volume Comparison Theorem, we see that

$$\mathcal{H}_m(B_w(r) \cap W_i) = \mathcal{H}_m(B_w(r) \cap U_w) \le \omega_m r^m \qquad \forall r \le r_i \tag{265}$$

is positive and finite for any $w \in W_i$. By Bishop-Gromov's Volume Comparison Theorem,

$$\frac{\mathcal{H}_m(B_w(r_1) \cap W_i)}{\mathcal{H}_m(B_w(r_2) \cap W_i)} \ge \frac{r_1^m}{r_2^m} \qquad \forall\ w \in W_i,\ r_1 < r_2 \le r_i. \tag{266}$$

Since W_i is a subset of the compact $F_{\delta_{i+1}}(Y^{\delta_{i+1}})$ it is precompact. Let $w_1, ...w_N \subset W_i$ be a maximal collection such that $B(w_i, r_i/2)$ are disjoint. Then

$$W_i \subset \bigcup_{n=1}^{N} B(w_n, r_i) \tag{267}$$

and

$$\mathcal{H}^m(W_i) \leq \sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i) \leq (1/4)^m \sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i/4)). \tag{268}$$

But it is not hard to see examining (209) that $B(w_n, r_i/2)$ are the limits of disjoint balls in M_j, so

$$\sum_{n=1}^{N} \mathcal{H}^m(B(w_n, r_i/4)) \leq \limsup_{j\to\infty} \mathcal{H}_m(M_j^{\delta_i}) \leq V_0. \tag{269}$$

So W_i has Hausdorff dimension m and

$$\mathcal{H}_m(W_i) \leq V_0. \tag{270}$$

Now

$$Y = \bigcup_{i=1}^{\infty} W_i \tag{271}$$

so it has Hausdorff dimension m and

$$\mathcal{H}_m(Y) \leq V_0. \tag{272}$$

Now to see that Y has positive density everywhere, we must show

$$\Theta_*(y, \mathcal{H}^m) = \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y, r))}{r^m} > 0. \tag{273}$$

For fixed $i \geq I_y$, we have

$$\mathcal{H}_m(B(y, r)) \geq \mathcal{H}_m(B(y, r) \cap W_i). \tag{274}$$

Combining this with (266) we have

$$\begin{aligned}
\Theta_*(y, \mathcal{H}^m) &= \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y, r) \cap W_i)}{r^m} && (275)\\
&\geq \liminf_{r\to 0} \frac{\mathcal{H}_m(B(y, r_i) \cap W_i)}{r_i^m} && (276)\\
&\geq \frac{\mathcal{H}_m(B(y, r_i) \cap W_i)}{r_i^m} > 0. && (277)
\end{aligned}$$

□

Open Question 8.9. *Are the glued limit spaces of sequences as in Theorem 8.8 unique?*

Open Question 8.10. *Are the glued limit spaces of sequences as in Theorem 8.8 countably $\mathcal{H}^m$ rectifiable?*

References

[1] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[2] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.

[3] Tobias H. Colding. Ricci curvature and volume convergence. *Ann. of Math. (2)*, 145(3):477–501, 1997.

[4] Tobias Holck Colding and Aaron Naber. Characterization of tangent cones of noncollapsed limits with lower ricci bounds and applications. *preprint*, 2011.

[5] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[6] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, 53:53–73, 1981.

[7] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.

[8] W. Klingenberg. Contributions to Riemannian geometry in the large. *Ann. of Math. (2)*, 69:654–666, 1959.

[9] Shigeru Kodani. Convergence theorem for Riemannian manifolds with boundary. *Compositio Math.*, 75(2):171–192, 1990.

[10] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.*, 353(9):3585–3602 (electronic), 2001.

[11] Jeremy Wong. An extension procedure for manifolds with boundary. *Pacific J. Math.*, 235(1):173–199, 2008.

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